Hawking radiation from dynamical horizons*

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Abstract

In completely local settings, we establish that a dynamically evolving spherically symmetric black hole horizon can be assigned a Hawking temperature and with the emission of flux, radius of the horizon shrinks.

The laws of black hole mechanics in general relativity are remarkably analogous to the laws of thermodynamics [1]. This analogy is exact when quantum effects are taken into account. Indeed, Hawking's semiclassical analysis establishes that quantum mechanically, a stationary black hole with surface gravity κ radiates particles to infinity with a perfect black body spectrum at temperature $\kappa/2\pi$ [2]. Consequently, asymptotic observers perceive a thermal state and assign a physical temperature to the black hole. The precise match to thermodynamics is complete when the thermodynamic entropy of the black hole is identified with a quarter of its area [3].

Although the proofs that have been provided over the years are elegant, they are quite restrictive, inapplicable even for spacetimes with superradiance [4]. These formulations also do not indicate how such a thermal state may arise as a result of some version of physical process. In addition, it seems to be a reasonable physical expectation that even with a local definition of black hole horizon one should be able to establish the analogy to thermodynamics. More precisely, such horizons should have a temperature of $\kappa/2\pi$. Incidentally, this question has been investigated in a semiclassical approach which treats Hawking radiation as a quantum tunneling phenomenon [5, 6]. Still there are some problems with the method itself and some issues which have not been addressed in this treatment of dynamical horizons and it is not clear how the horizon loses area due to emission of a flux of radiation.

In this paper, a formalism is developed to establish two basic issues (see [7] for details). First, that one can associate a temperature to local dynamical horizons without the need of any WKB-like approximations. Second, that there exists a precise relation between the radiation emitted by the horizon and area loss, i.e., flux of outgoing radiation through the horizon in between two partial Cauchy slices exactly equals the difference of radii of the sphere that foliates the horizon at those two instances.

We begin with definitions. We follow the conventions of [8]. Consider a four dimensional spacetime $\mathscr{M}$ with signature $(-,+,+,+)$. A three-dimensional submanifold Δ in $\mathscr{M}$ is said to be a *future outer trapping horizon* (FOTH) if `1)` It is foliated by a preferred family of topological two-spheres such that, on each leaf S, the expansion θ_+ of a null normal l_+^a vanishes and the expansion θ_- of the other null normal l_-^a is negative definite, `2)` The directional derivative of

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θ_+ along the null normal l_-^a (i.e., $\mathscr{L}_{l\_}\theta_+$) is negative definite. Thus, Δ is foliated by marginally trapped two-spheres. According to a theorem due to Hawking, the topology of S is necessarily spherical in order that matter or gravitational flux across Δ is non-zero. If these fluxes are identically zero then Δ becomes a Killing or isolated horizon. Even though our arguments will remain local, for definiteness, we choose a spherically symmetric background metric

$$ds^2 = -2e^{-f}dx^+dx^- + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

where both f and r are smooth functions of $x^\pm$. The expansions of the two null normals are $\theta_\pm = (2/r)\,\partial_\pm r$ respectively where $\partial_\pm = \partial/\partial x^\pm$. In this coordinate system, the second requirement for FOTH translates to $\partial_-\theta_+ < 0$ on Δ.

Let the vector field $t^a = l_+^a + h\,l_-^a$ be tangential to the FOTH for some smooth function h. Then the Raychaudhuri equation for l_+^a and the Einstein equation implies

$$\partial_+\theta_+ = -h\partial_-\theta_+ = -8\pi\, T_{++}. \tag{2}$$

where $T_{++} = T_{ab}\, l_+^a l_+^b$ and T_{ab} is the energy momentum tensor. Since $t^2 = -2h\, e^{-f}$, a FOTH becomes spacelike if and only if $T_{++} > 0$ and is timelike if and only if $T_{++} < 0$. For a timelike FOTH, several consequences follow. Here, $\mathscr{L}_t r < 0$, and hence, Δ is timelike if and only if the area A and the Misner-Sharp energy E decreases along the horizon. This is also expected on general grounds since the horizon receives an incoming flux of negative energy, $T_{++} < 0$.

In the dynamical spacetime (1) the Kodama vector field plays the analog role of the Killing vector [9]. For this spacetime, it is given by

$$K^a = e^f\,(\partial_- r)\,\partial_+^a - e^f\,(\partial_+ r)\,\partial_-^a. \tag{3}$$

The surface gravity is defined through $K^a\nabla_{[b}K_{a]} = \kappa\, K_b$ and is $\kappa = -e^f\,\partial_-\partial_+ r$. The FOTH condition $\partial_-\theta_+ < 0$ implies $\kappa > 0$. We determine the positive frequency modes of the Kodama vector. It is given by

$$Z_\omega = F(r)\begin{cases} \theta_+^{-\frac{i\omega}{\kappa}} & \text{for } \theta_+ > 0 \\ (|\theta_+|)^{-\frac{i\omega}{\kappa}} & \text{for } \theta_+ < 0. \end{cases} \tag{4}$$

where the spheres are not trapped 'outside the trapping horizon' ($\theta_+ > 0$) and fully trapped 'inside' ($\theta_+ < 0$). These are precisely the modes which are defined outside and inside the dynamical horizon respectively but not on the horizon. Now we have to keep in mind that the modes (4) are not ordinary functions, but are distribution-valued. Using the standard results [10], we find for

$$(\theta_+ + i\epsilon)^\lambda = \begin{cases} \theta_+^\lambda & \text{for } \theta_+ > 0 \\ |\theta_+|^\lambda e^{i\lambda\pi} & \text{for } \theta_+ < 0 \end{cases} \tag{5}$$

for the choice $\lambda = -i\omega/\kappa$. For spherically symmetric static case, see [11]. The distribution (5) is well-defined for all values of θ_+ and λ, and it is differentiable to all orders. The modes Z_ω^* are given by the complex conjugate distribution.

We calculate the probability density in a single particle Hilbert space for positive frequency solutions across the dynamical horizon and is given by, apart from a positive function of r,

$$\begin{aligned} \varrho(\omega) &= \omega(\theta_+ + i\epsilon)^{-\frac{i\omega}{\kappa}}(\theta_+ - i\epsilon)^{\frac{i\omega}{\kappa}}. \\ &= \begin{cases} \omega & \text{for } \theta_+ > 0 \\ \omega e^{\frac{2\pi\omega}{\kappa}} & \text{for } \theta_+ < 0. \end{cases} \end{aligned} \tag{6}$$

The conditional probability that a particle emits when it is incident on the horizon from inside is,

$$P_{(emission|incident)} = e^{-\frac{2\pi\omega}{\kappa}} \tag{7}$$

This gives the correct Boltzmann weight with the temperature $\kappa/2\pi$, which is the desired value.

We now show that as the horizon evolves, the radius of the 2-sphere foliating the horizon shrinks in precise accordance with the amount of flux radiated by the horizon. The line-element (1) induces a line-element on Δ

$$ds^2 = -2e^{-f}h^{-1}(d\tilde{x}^-)^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{8}$$

Consequently, the volume element on the FOTH is given by $d\mu = \sqrt{2e^{-f}h^{-1}}r^2 \sin\theta \, d\tilde{x}^- d\theta d\phi$. We can now calculate the flux of matter energy that crosses the dynamical horizon—it is an integral on a slice of horizon bounded by two spherical sections S_1 and S_2

$$\mathscr{F} = \int d\mu \; T_{ab}\hat{n}^a K^b \tag{9}$$

where $\hat{n}^a$ is the unit normal vector

$$\hat{n}^a = \frac{1}{\sqrt{2he^{-f}}}(\partial_+^a - h\partial_-^a) \tag{10}$$

and K^a is the Kodama vector. Using spherical symmetry, and the Einstein equations, we get (see [7] for details)

$$\mathscr{F} = -\frac{1}{2}(r_2 - r_1) \tag{11}$$

where r_1, r_2 are respectively the two radii of S_1, S_2. Since the area is decreasing along the horizon, $r_2 < r_1$ where S_2 lies in the future of S_1. As a result, the outgoing flux of matter energy radiated by the dynamical horizon is positive definite (and the ingoing flux of matter energy is negative definite). The flux formula (11) differs from the one given in [12] in an important way: since the Kodama vector field provides a timelike direction and is null on the horizon, it is appropriate to use K^a in the flux formula for the dynamical horizon.

Our derivation of Hawking temperature and the flux law depends on two assumptions. The first is the existence of the Kodama vector field and the Misner-Sharp energy. For spherically symmetric spacetimes, the Kodama vector field exists unambiguously and the Misner-Sharp energy is well defined. For more general spacetimes, a Kodama-like vector field is not known, however, one can still define some mass for such cases that reduces to the Misner-Sharp energy in the spherical limit. The second assumption, is the slow variation of the dynamical surface gravity κ during evolution. For large black holes, the horizon evolves slowly enough so that the surface gravity function should vary slowly in some small neighbourhood of the horizon. In other words, this derivation implies that the Hawking temperature for a dynamically evolving large black hole is $\kappa/2\pi$ if the dynamical surface gravity is slowly varying in the vicinity of the horizon.

It is also interesting to speculate on the extension of the present method for other diffeomorphism invariant theories of gravity. While the zeroth and the first law hold for any arbitrary such theory, the second law has only been proved for a class of such theories [13]. If the present formalism can be extended to other theories of gravity, it will lend a support to the existence of the area increase theorem for such theories. While more interesting and deeper issues can only be understood in a full quantum theory of gravity, the present framework can provide a better understanding of the Hawking radiation process.

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