Two dimensional hydrodynamics with gauge and gravitational anomalies

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References


Modern applications of the chiral anomaly

1. Quantum wires
2. Quantum Hall effect
3. Hawking effect
4. Chiral magnetic effect
5. Chiral vortical effect
6. Anomalous hydrodynamics
Relativistic Fluid Dynamics

▶ Necessity:
▶ Large velocity (comparable to light) of macroscopic flow.
▶ Microscopic motion of fluid particles is large.
Relativistic Fluid Dynamics

- **Necessity:**
  - Large velocity (comparable to light) of macroscopic flow.
  - Microscopic motion of fluid particles is large.

- **Equation of Motion:**

\[
\partial_\mu T_{\nu}^{\mu} = 0
\]

- Conservation of Energy-momentum tensor.
- For a charged fluid this is supplemented with

\[
\partial_\mu J^{\mu} = 0
\]
Constitutive Relations:

- Additional relations expressing E.M tensor/Charge in terms of the basic fluid variables like velocity, temperature and chemical potential.
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- **Ideal Fluid Relations:**

\[
T_{\mu\nu} = (\varepsilon + P) u_{\mu\nu} + P \eta_{\mu\nu},
\]

\[
J_\mu = n u_\mu
\]

- $\varepsilon \rightarrow$ energy density, $P \rightarrow$ pressure, $n \rightarrow$ charge density,
  $\eta_{\mu\nu} \rightarrow$ metric, $u_\mu \rightarrow$ fluid velocity normalised as $u^\mu u_\mu = -1$.

- Extra terms have to be included in the non ideal case to include effects of dissipation (like viscosity).
Two Approaches

- **Landau type approach:**
  Constitutive relations are derived to ensure positivity of entropy and hence compatibility with a local version of the second law of thermodynamics. Also, it satisfies the appropriate equations of motion.
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▶ **Derivative expansion approach:**
Effective action is expressed as a series in powers of derivatives acting on fluid variables (like velocity). This is the large wavelength approximation.
Likewise, the constitutive relations are also expressed as a power series.
Results from these approaches agree although a general proof of this statement is missing.
Hydrodynamics in presence of gauge/gravity

Turn on a gauge field \((A_\mu)\) and gravity (metric \(g_{\mu\nu}\)).

**Changes**
Replace ordinary derivative by covariant derivative in the conservation laws

\[
D_\mu T^\mu_\nu = 0, \quad D_\mu J^\mu = 0;
\]

Modify constitutive relations:
Depend on gauge and/or diffeomorphism invariant combinations of the fields \((A_\mu, g_{\mu\nu})\).
What happens if anomalies are present?

\[
D_\mu T^\mu_\nu \neq 0, \quad D_\mu J^\mu \neq 0;
\]

A hydrodynamic (derivative) expansion is usually adopted
Review on anomalies

- **Standard definition**
  Anomaly is the breakdown of a classical symmetry upon quantization.
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  Anomaly is the breakdown of a classical symmetry upon quantization.

- **Example:**
  QED

  \[ \partial_\mu J^\mu = 0, \quad \partial_\mu J^{\mu 5} = 0; \]

  Both vector/axial vector currents are conserved. Results follow on using the classical equation of motion (Noether’s theorem).

  More refined calculation yields,

  \[ \partial_\mu J^\mu = 0, \quad \partial_\mu J^{\mu 5} = \frac{1}{16\pi^2} \epsilon_{\mu \nu \alpha \beta} F^{\mu \nu} F^{\alpha \beta}; \]
Infinities and Anomalies

Anomaly is the breakdown of formal manipulations (ignoring infinities) leading to a modified conservation law.
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▶ Same example (QED):

\[ \partial_{\mu} J^{\mu}(x) = \partial_{\mu} (\bar{\psi} \gamma^{\mu} \psi) = \bar{\psi} \not{\partial} \psi + \psi \not{\partial} \bar{\psi} \]

▶ Classical equations of motion

\[ \not{\partial} \psi = m\psi, \quad \not{\partial} \bar{\psi} = -m\bar{\psi} \]

\[ \partial_{\mu} J^{\mu}(x) = m\bar{\psi}(x)\psi(x) - m\bar{\psi}(x)\psi(x) = 0 \]
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Allowed only if \( \bar{\psi}(x)\psi(x) \) is not infinity!

▸ Fields at identical space-time points not well defined and could lead to infinities.
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- **Same example (QED):**

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\]

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\]

\[
\partial_\mu J^\mu (x) = m \bar{\psi}(x) \psi(x) - m \bar{\psi}(x) \psi(x)
\]

\[= 0\]

Allowed only if \(\bar{\psi}(x) \psi(x)\) is not infinity!

- **Fields at identical space-time points not well defined and could lead to infinities.**

\[
\langle T \bar{\psi}(x) \psi(y) \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{i p \cdot (x-y)}}{\not{p} - m}
\]

- for \(x=y\) \(\int \frac{d^4 p}{(2\pi)^4} \frac{1}{\not{p} - m} \to \text{divergent at } p \to \infty\)
Chiral anomaly

Anomaly in the chiral current

\[ \partial_\mu \left[ \bar{\psi} \gamma^\mu \left( \frac{1 \pm \gamma^5}{2} \right) \psi \right] = A \]

No regularisation exists for which \( A = 0 \)
Chiral anomaly

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$$\partial_\mu \left[ \bar{\psi} \gamma^\mu \left( \frac{1 \pm \gamma^5}{2} \right) \psi \right] = A$$

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- Covariant and consistent anomaly

- Current transforming covariantly under a gauge transformation is called covariant current.
  Anomaly of a covariant current also transforms covariantly $\rightarrow$ Covariant anomaly.
Chiral anomaly

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**Covariant and consistent anomaly**

- Current transforming covariantly under a gauge transformation is called covariant current. Anomaly of a covariant current also transforms covariantly \( \rightarrow \) Covariant anomaly.
- Current defined from the variation of an effective action is called consistent current. Anomaly of consistent current is consistent anomaly (satisfies W-Z consistency condition).

Covariant and consistent expressions are complementary, related by local polynomials.
Example from 2 dimensions (chiral gauge anomaly)

Covariant anomaly

\[ \partial_{\mu} J^{\mu} = \frac{1}{4\pi} \epsilon_{\mu\nu} F^{\mu\nu} \]

- Effective action

\[ W[A] = \int_{0}^{1} dg \int d^2 x \ A_\mu(x) J^{\mu(g)}(x) \]

Formal definition is made sensible by specifying a regularisation for \( J^{\mu(g)}(x) \).
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Formal definition is made sensible by specifying a regularisation for \( J^{\mu(g)}(x) \).

- **Choose a gauge invariant regularisation**

\[ W[A - \partial \alpha] = \int_0^1 dg \int d^2 x \; (A_\mu - \partial_\mu \alpha) J^{\mu(g)} \]

\[ \int d^2 x \left( \partial_{\mu} \frac{\delta W}{\delta A_\mu} \right) \alpha = \int_0^1 dg \int d^2 x \; \alpha(x) \partial_\mu J^{\mu(g)} \]

\[ \partial_{\mu} \frac{\delta W}{\delta A_\mu} = \int_0^1 dg \left( \frac{g}{4\pi} \epsilon_{\mu\nu} F^{\mu\nu} \right) = \frac{1}{8\pi} \epsilon_{\mu\nu} F^{\mu\nu} \]

This is the consistent anomaly.
Example continued

Covariant: \( \frac{1}{4\pi} \epsilon_{\mu\nu} F^{\mu\nu} \); Consistent: \( \frac{1}{8\pi} \epsilon_{\mu\nu} F^{\mu\nu} \)

- For any d=2n dimensions

  Consistent anomaly = \( \frac{1}{n+1} \) Covariant anomaly

  Follows from homogeneous nature of anomaly
Example continued

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- Relation between covariant and consistent currents

$$J^{Cov}_\mu = J^{Const}_\mu + \frac{1}{4\pi} \epsilon_{\mu\nu} A^\nu$$
Example continued

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- Relation between covariant and consistent currents

$$J^{Cov}_\mu = J^{Const}_\mu + \frac{1}{4\pi} \epsilon_{\mu\nu} A^\nu$$

- Extra piece (local polynomial) does not contribute to the effective action.

$$\int_0^1 dg \int d^2 x \ A_\mu \left( \frac{1}{4\pi} \right) \epsilon^{\mu\nu} A_\nu = 0$$

$W$ is therefore unaffected by the regularisation prescription.
Gravitational anomaly (2 dimensions)

- Occurs in (4n-2) dimension. (2,6,10,..)

- For a usual (non-chiral) theory, one can trade between the conformal (trace) and general coordinate (diffeomorphism) symmetries.

\[ T_{\mu}^{\mu} = 0, \ \nabla_{\mu} T_{\nu}^{\mu} \neq 0 \ \text{OR} \ \ T_{\mu}^{\mu} \neq 0, \ \nabla_{\mu} T_{\nu}^{\mu} = 0 \]
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- For a chiral theory, both conformal and general coordinate invariances are broken

\[ T_{\mu}^{\mu} \neq 0, \quad \nabla_\mu T_{\nu}^{\mu} \neq 0 \]

- As in the gauge theory, here also there are covariant and consistent expressions for the diffeomorphism anomaly.
Anomalous Ward identities

- Diffeomorphism anomaly:

\[ \nabla_\nu T^{\nu \mu} = F_\nu^\mu J^\nu + C_g \bar{\epsilon}^{\mu \nu} \nabla_\nu R, \]
Anomalous Ward identities

- **Diffeomorphism anomaly:**

  \[ \nabla_\nu T^{\nu \mu} = F^\mu_\nu J^\nu + C_g \bar{\epsilon}^{\mu \nu} \nabla_\nu R, \]

- **Conformal anomaly:**

  \[ T^\mu_\mu = C_w R, \]
Anomalous Ward identities

- **Diffeomorphism anomaly:**

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- **Gauge anomaly:**

\[ \nabla_\mu J^\mu = C_s \epsilon^{\mu \nu} F_{\mu \nu}. \]
Anomalous Ward identities

- **Diffeomorphism anomaly:**
  \[ \nabla_{\nu} T^{\nu\mu} = F^{\mu\nu} J^{\nu} + C g \bar{\epsilon}^{\mu\nu} \nabla_{\nu} R, \]

- **Conformal anomaly:**
  \[ T^{\mu}_{\mu} = C w R, \]

- **Gauge anomaly:**
  \[ \nabla_{\mu} J^{\mu} = C s \bar{\epsilon}^{\mu\nu} F_{\mu\nu}. \]

Follows from purely algebraic arguments. \( J^{\mu}, T^{\mu\nu} \) are covariant current/stress tensor, \( R \) is the Ricci scalar, \( F_{\mu\nu} = \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu} \) is field strength
General set up (1+1 dimensional static space-time)

- Metric:

\[ ds^2 = -e^{2\sigma(r)} dt^2 + g_{11}(r) dr^2 \]
General set up (1+1 dimensional static space-time)

- **Metric:**
  
  \[ ds^2 = -e^{2\sigma(r)} dt^2 + g_{11}(r) dr^2 \]

- **Null Coordinates:**
  \[ u = t - r^*, \quad v = t + r^* \]
  
  \[ \frac{dr}{dr^*} = -\frac{e^{\sigma}}{\sqrt{g_{11}}} \]
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- **Notation:** \( g = det g_{\mu\nu} = -\frac{e^{4\sigma}}{4} \)
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- **Antisymmetric tensor:**

\[
\bar{\epsilon}_{\mu\nu} = \sqrt{-g} \epsilon_{\mu\nu} = \frac{e^{2\sigma}}{2} \epsilon_{\mu\nu}
\]

\[
\epsilon_{uv} = -\epsilon_{vu} = 1
\]
Passage to hydrodynamics

Introduce the velocity $u^\mu$ of the time independent equilibrium fluid fields, satisfying $u^\mu u_\mu = -1$ (comoving frame)

$$u^\mu = e^{-\sigma(r)}(1,0), \quad u_\mu = - e^{\sigma(r)}(1,0), \quad (\mu = t, r)$$
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- **Null Coordinates**

$$u_\mu = -\frac{e^{\sigma(r)}}{2}(1,1), \quad u^\mu = e^{-\sigma(r)}(1,1), \quad (\mu = u, v)$$

- **Dual vector**

$$\tilde{u}_\mu = \epsilon_{\mu \nu} u^\nu = \frac{e^{\sigma(r)}}{2}(1, -1)$$

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normalised as $\tilde{u}^\mu \tilde{u}_\mu = 1$
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normalised as $\tilde{u}^\mu \tilde{u}_\mu = 1$

- **Chiral vector**

$$u_\mu^c = u_\mu - \tilde{u}_\mu = -\bar{\epsilon}_{\mu\nu} u^{\nu c}$$
Passage to hydrodynamics

- Chemical potential

\[ \mu = A_t(r)/\sqrt{-g_{00}} = A_t(r)e^{-\sigma} \]
Passage to hydrodynamics

- **Chemical potential**

\[ \mu = A_t(r) / \sqrt{-g_{00}} = A_t(r) e^{-\sigma} \]

- **U(1) gauge field**

\[ A_\mu = (A_t(r), 0) \]
Passage to hydrodynamics

- **Chemical potential**
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- **Temperature**
  \[ T = T_0 e^{-\sigma} \]

where \( T_0 \) is the equilibrium temperature
Passage to hydrodynamics

▶ **Chemical potential**

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▶ **U(1) gauge field**

\[ A_\mu = (A_t(r), 0) \]

▶ **Temperature**

\[ T = T_0e^{-\sigma} \]

where \( T_0 \) is the equilibrium temperature

▶ **Ricci scalar**

\[ R = \frac{1}{g_{11}^2}(g_{11}'\sigma' - 2g_{11}\sigma'^2 - 2g_{11}\sigma'') = -2u^\mu \nabla^\nu \nabla_\mu u_\nu \]
Constitutive relations

- Energy-momentum Tensor:

\[
T_{\mu\nu} = \left[ C_1 T^2 - C_w \left( u^\alpha \nabla^\beta \nabla_\beta u_\alpha \right) + \mu^2 \left( \frac{1}{2\pi} - C_s \right) \right] g_{\mu\nu}
\]
\[
+ \left[ 2C_w \left( u^\alpha \nabla^\beta - u^\beta \nabla^\alpha \right) \nabla_\alpha u_\beta + 2C_1 T^2 + 2\mu^2 \left( \frac{1}{2\pi} - C_s \right) \right] u_\mu u_\nu
\]
\[
- \left[ 2C_g \left( u^\alpha \nabla^\beta - u^\beta \nabla^\alpha \right) \nabla_\alpha u_\beta + C_2 T^2 + C_s \mu^2 \right] (u_\mu \tilde{u}_\nu + \tilde{u}_\mu u_\nu)
\]
\[
+ \left\{ \left( \frac{C}{\pi} - 2(C + P)C_s \right) \frac{T}{T_0} \mu + \left( \frac{C^2 + P^2}{2\pi} - C_s (C + P)^2 \right) \frac{T^2}{T_0^2} \right\} (2u_\mu u_\nu + g_{\mu\nu})
\]
\[
+ \left\{ \left( \frac{P}{\pi} - 2(C + P)C_s \right) \frac{T}{T_0} \mu + \left( \frac{CP}{\pi} - C_s (C + P)^2 \right) \frac{T^2}{T_0^2} \right\} (u_\mu \tilde{u}_\nu + \tilde{u}_\mu u_\nu)
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Constitutive relations

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T_{\mu\nu} = [C_1 T^2 - C_w (u^\alpha \nabla^\beta \nabla_\beta u_\alpha) + \mu^2 \left( \frac{1}{2\pi} - C_s \right)] g_{\mu\nu} \\
+ [2 C_w (u^\alpha \nabla^\beta - u^\beta \nabla^\alpha) \nabla_\alpha u_\beta + 2 C_1 T^2 + 2 \mu^2 \left( \frac{1}{2\pi} - C_s \right)] u_\mu u_\nu \\
- [2 C_g (u^\alpha \nabla^\beta - u^\beta \nabla^\alpha) \nabla_\alpha u_\beta + C_2 T^2 + C_s \mu^2] (u_\mu \tilde{u}_\nu + \tilde{u}_\mu u_\nu) \\
+ \left\{ \left( \frac{C}{\pi} - 2(C + P)C_s \right) \frac{T}{T_0} \mu + \left( \frac{C^2 + P^2}{2\pi} - C_s (C + P)^2 \right) \frac{T^2}{T_0^2} \right\} (2 u_\mu u_\nu + g_{\mu\nu}) \\
+ \left\{ \left( \frac{P}{\pi} - 2(C + P)C_s \right) \frac{T}{T_0} \mu + \left( \frac{CP}{\pi} - C_s (C + P)^2 \right) \frac{T^2}{T_0^2} \right\} (u_\mu \tilde{u}_\nu + \tilde{u}_\mu u_\nu)
\]

- **Gauge current**

\[
J_\mu = -2 C_s \mu (u_\mu + \tilde{u}_\mu) + \frac{\mu}{\pi} u_\mu + \left( \frac{C}{\pi} - 2(C + P)C_s \right) \frac{T}{T_0} u_\mu \\
+ \left( \frac{P}{\pi} - 2(C + P)C_s \right) \frac{T}{T_0} \tilde{u}_\mu,
\]

Here, \(C_1, C_2, P\) and \(C\) are constants.
Derivative expansion approach

- **Covariant stress tensor**

\[ T^\mu\nu = \varepsilon u^\mu u^\nu + \mathcal{P}\tilde{u}^\mu \tilde{u}^\nu + \theta (\tilde{u}^\mu u^\nu + u^\mu \tilde{u}^\nu) \]

- General form of a symmetric second rank tensor constructed from \( u_\mu \) and \( \tilde{u}_\mu \).
Derivative expansion approach

- **Covariant stress tensor**

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T^{\mu\nu} = \varepsilon u^\mu u^\nu + \mathcal{P} \tilde{u}^\mu \tilde{u}^\nu + \theta (\tilde{u}^\mu u^\nu + u^\mu \tilde{u}^\nu)
\]

- General form of a symmetric second rank tensor constructed from \(u_\mu\) and \(\tilde{u}_\mu\).

\[
\varepsilon = C_1 T^2 + C_w \left(u^\nu \nabla^\mu \nabla_\mu u_\nu\right) + 2C_w \left(u^\mu \nabla^\nu - u^\nu \nabla^\mu\right) \nabla_\mu u_\nu
\]
\[
\mathcal{P} = C_1 T^2 - C_w \left(u^\nu \nabla^\mu \nabla_\mu u_\nu\right)
\]
\[
\theta = -C_2 T^2 - 2C_g \left(u^\mu \nabla^\nu - u^\nu \nabla^\mu\right) \nabla_\mu u_\nu
\]

\(C_1\) and \(C_2\) are undetermined parameters expressed in terms of the normalisation factors \((C_w, C_g)\) of the trace/diffeomorphism anomalies.
Derivative expansion approach

- **Covariant stress tensor**

\[ T^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} + \mathcal{P} \tilde{u}^{\mu} \tilde{u}^{\nu} + \theta (\tilde{u}^{\mu} u^{\nu} + u^{\mu} \tilde{u}^{\nu}) \]

- General form of a symmetric second rank tensor constructed from \( u_\mu \) and \( \tilde{u}_\mu \).

\[ \varepsilon = C_1 T^2 + C_w (u^{\nu} \nabla^\mu \nabla_\mu u_\nu) + 2C_w (u^{\mu} \nabla^\nu - u^{\nu} \nabla^\mu) \nabla_\mu u_\nu \]

\[ \mathcal{P} = C_1 T^2 - C_w (u^{\nu} \nabla^\mu \nabla_\mu u_\nu) \]

\[ \theta = -C_2 T^2 - 2C_g (u^{\mu} \nabla^\nu - u^{\nu} \nabla^\mu) \nabla_\mu u_\nu \]

- \( C_1 \) and \( C_2 \) are undetermined parameters expressed in terms of the normalisation factors \((C_w, C_g)\) of the trace/diffeomorphism anomalies. Non-trivial relations,

\[ C_1 = 4\pi^2 C_w, \quad C_2 = 8\pi^2 C_g \]
Israel-Hartle-Hawking condition

- Defined by taking $T_{\mu \nu}/J_{\mu}$ in Kruskal coordinates, corresponding to both outgoing and ingoing modes, as regular, near the horizon.
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- Implies $T_{uu}, T_{vv}, J_u, J_v \to 0$ near the horizon

- Horizon ($r = r_0$) defined as $e^{2\sigma}|_{r_0} = \frac{1}{g_{11}}|_{r_0} = 0$
Israel-Hartle-Hawking condition

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- Horizon ($r = r_0$) defined as $e^{2\sigma}|_{r_0} = \frac{1}{g_{11}}|_{r_0} = 0$

- Fixes all the undetermined constants:

  C and P fixed from $J_u, J_v \to 0$

  For $J_u \to 0$, $P - C = \mu e^\sigma|_{r_0} = 0$

  For $J_v \to 0$, $P + C = -\mu e^\sigma|_{r_0} = 0$

  \[ P=C=0; \]
Israel-Hartle-Hawking condition

- Defined by taking $T_{\mu\nu}/J_{\mu}$ in Kruskal coordinates, corresponding to both outgoing and ingoing modes, as regular, near the horizon.

- Implies $T_{uu}, T_{vv}, J_{u}, J_{v} \to 0$ near the horizon

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- $C_1$ and $C_2$ fixed from the condition on stress tensor.

  $$C_1 = 4\pi^2 C_w, \quad C_2 = 8\pi^2 C_g,$$
Response parameters and anomaly coefficients

\[ J_\mu = -2C_s \mu (u_\mu + \tilde{u}_\mu) + \frac{\mu}{\pi} u_\mu \]
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- \( s_2, s_4 \) are certain combinations of gauge field that occur in the second order expansion, and,

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- Comparison yields

\[ \frac{\partial P}{\partial \mu} = T^2 \frac{\partial p_0}{\partial \mu} = \left( -2C_s + \frac{1}{\pi} \right) \mu; \quad a_2 = a'_2 = 0 \]
Final Expressions

\[ p_0 = \left( \frac{1}{\pi} - C_s \right) \frac{\mu^2}{T^2} + Q(int.\,const) \]

- In the absence of gauge field
  \[ p_0 = C_1 = 4\pi^2 C_w \]

- General solution:
  \[ p_0 = 4\pi^2 C_w + \left( \frac{1}{\pi} - C_s \right) \frac{\mu^2}{T^2} \]

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- **Consistency check**

The constitutive relation for \( T_{\mu\nu} \) agrees with the form obtained by the derivative expansion provided the above identifications are used.
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Thank You