SYSTEMATIC STUDIES ON CLASSICAL AND QUANTUM DYNAMICAL SYSTEMS

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions. The work was done under the guidance of Prof. S. C. Mishra, Department of Physics, Kurukshetra University, Kurukshetra, and Dr. C. Nagaraja Kumar Department of Physics, Panjab University, Chandigarh.

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Thesis

Dedicated to my parents,
who constantly inspire me to acquire
knowledge on science

Contents

Abstract

Integrability has always played a significant role in the qualitative study of classical and quantum dynamical systems. A real Hamiltonian function $H(x, p, t)$, with convenient combination of the functions of canonical pairs, say coordinate x and momentum p in the form of kinetic and potential energies, explains a variety of physical properties of the concerned dynamical system. The Hamiltonian formulation of dynamics is the mathematically most beautiful form of mechanics and (in fact) the stepping stone to classical and quantum mechanics. Hamiltonian systems are conserved dynamical systems with a very interesting algebraic structure in the guise of the Poisson bracket. Hamilton's equations are invariant under a very wide class of transformation (the canonical transformations), and this leads to a number of powerful solutions, which further gave many new examples and some good techniques with the development of integrable systems.

A classical Hamiltonian $H(x, p)$ is a function from a $2n$-dimensional phase space into the real numbers. Hamiltonian theory is an important element of integrable systems, whether discrete, ordinary differential or partial differential equations. The concept of classical integrability was first precisely stated by Liouville, who defined that a classical Hamiltonian $H(x, p)$ is integrable when it possesses n functionally independent first integrals in involution with a certain degree of regularity, e.g., being smooth or analytic. The complexity of the dynamics defined by H, i.e., the regular or chaotic behavior of the orbits of the Hamiltonian, strongly depends upon its integrability. Particularly, the dynamics is not considered chaotic when H is integrable.

In the classical theory (Lie, Liouville, etc.) of ordinary differential equations there are remarkable results which relate the property of integrability of ODEs in quadratures with the existence of continuous symmetries and first integrals. Symmetries and first integrals may serve as a solid mathematical foundation for an algebraic theory of integrable equations. In the case of integrable partial differential equations (PDEs) such a theory does already exist and proved to be extremely efficient. The key property of integrable PDEs is the existence of infinite hierarchies of local infinitesimal symmetries generated by a recursion operator. Characteristic features of integrable Hamiltonian PDEs are multi-Hamiltonian structures and hierarchies of local conservation laws. Study of these structures enable us to formulate constructive and very efficient tests for integrability and even solve the classification problem for some classes of equations.

In mathematical physics, one usually deals with concrete classical and quantum models (with a possible

dependence on free parameters) and tries to compute first integrals of the Hamiltonian explicitly with the aid of group theory. The notion of complete integrability goes back to the celebrated Liouvilles theorem for a system with a finite number of degrees of freedom. A quantum Hamiltonian $H(x;p)$ is a self-adjoint operator acting on a separable Hilbert space H. The naive definition of quantum integrability, which appears in many research papers, stems from that of classical integrability. Namely, a quantum Hamiltonian $H(x,p)$ is integrable in this naive sense when there exist n independent linear operators $\Gamma_i(i = 1.......n)$ which commute among them and with the Hamiltonian, n being the dimension of the quantum system. The contrast and resemblance between classical and quantum mechanics and/or field theory has been a good source of stimulus for theoretical physicists since the inception of quantum theory at the beginning of the twentieth century. In spite of the well-publicized differences such as the instability (stability) of the hydrogen atom in classical (quantum) mechanics, the photo-electric effect, and tunneling effects, classical and quantum mechanics share many common theoretical structures (in particular, the canonical formalism) and under certain circumstances provide (almost) the same predictions, as exemplified by the correspondence principle and Ehrenfests theorem

Important new examples of completely integrable classical mechanical systems were discovered by using “soliton techniques”, such as Lax pairs and bi-Hamiltonian representations. However, a whole new theory of Poisson brackets for PDEs was developed, starting with the discovery of two Poisson bracket representations of the KdV equation. This is called the bi-Hamiltonian property and has been established for a large number of systems. It has become one of the signatures of integrability. Over the years large families of systems with two or more compatible Poisson brackets (known as multi-Hamiltonian in general) have been discovered.

In the quantum case, there are two main approaches to this topic, each one corresponding to a different area of physics. They are widely used to describe complex phenomena in various sciences such as fluid dynamics, condensed matter physics, biophysics, plasma physics, nonlinear optics, quantum field theory and particle physics. The construction of a quantum variant of a given dynamical model, known in classical formulation (quantization) is not unique. Integrable Systems are systems that, although highly nontrivial and nonlinear, are amenable to exact and rigorous techniques for their solvability. They can take many shapes or forms: nonlinear evolution equations, partial and ordinary differential equations and difference equations, Hamiltonian many-body systems, quantum systems and spin models in statistical mechanics. A large number of mathematical techniques have been developed to unravel the rich structures behind these systems.

As is well-known, a classical Hamiltonian system with finitely many degrees of freedom can be transformed into action-angle variables by quadrature if a complete set of involutive independent conserved quantities can be obtained. It is a good challenge to formulate a quantum counterpart of the ‘transformation into the action-angle variables by quadrature’. Liouville finally provided a general framework characterizing the cases where the equations of motion are “solvable by quadratures”. A quantum Liouville theorem on completely integrable systems previously found indeed pertained to this setting.

To study the underlying properties of a dynamical system, it is important to be know that whether the system is integrable, and if it is, we wants to know as many quantities as possible whose values are conserved during the time evolution of the equation of motion. The global quantity (invariant) mentioned is a functional from the space of dependent variables (phase space) to the real (or complex) numbers. In this context we explore invariants for different classical and quantum integrable systems for both TD and TID dynamical systems, We are also searching and applying traditional methods for construction of invariants for different dynamical systems. In the beginning of ninetieth century Ermakov technique was used to find invariants, so we are also studying different aspects of generalized systems. A complex form of Hamiltonian is expected to offer some clues about some additional physical properties, which remain disregarded in real form. Therefore, the study of associated complex dynamical invariants becomes desirable and if they exist and are available, can provide insight into the analysis of a physical problem. The study of the complex invariants for a dynamical system can further open new vistas at the level of both classical and quantum mechanics.

The complex Hamiltonian systems are those in which Hamiltonian is non-hermitian and corresponding eigen values and eigen functions are complex. A consistent physical theory of quantum mechanics can be built on a complex Hamiltonian that is non-hermitian but instead satisfies the physical condition of space-time reflection symmetry ($\mathcal{PT}$-symmetry).

A class of complex Hamiltonians called the $\mathcal{PT}$-symmetric Hamiltonian in which the Hamiltonian is non-Hermitian but eigenvalue is real and this reality of the spectrum is a consequence of the combined action of parity and time reversal invariance of H (hence the name $\mathcal{PT}$-symmetry). The complex versions of x and p using above symmetry transformation can be written as:

$$x = x_1 + ip_2; \quad p = p_1 + ix_2.$$

where (x_1, p_1) and (x_2, p_2) are real and considered as canonical pairs. In this way, one can transform a real phase space (x, p) into a complex phase space (x_1, p_2, p_1, x_2) with additional degrees of freedom namely x_2 and p_2. The complex phase space characterized by the equation for x and p are known as the extended complex phase space (ECPS).

In this approach both x and p are separately made complex by extending each of them to the corresponding complex planes, i.e. inserting an imaginary component in each. This transformation has been used by many researchers to complexify the coordinates for phase transformations. However, these studies are confined to one dimensional system. Not much attempts have been made to extend this approach in higher dimensions. The extension of such studies in higher dimensions is always desirable to explore the possibility of finding more applications of this method. With this motivation we have generalize extended complex approach in two dimensions and got some concrete results for complex dynamical systems.

Thesis is planned systematically as follows:

Survey and discussion on the classical and quantum dynamical systems is mentioned in chapter 1. Since integrability plays a major role in this study, we discuss the concept of integrability, different forms integrability, Hamiltonian systems and Liouville integrability. Their is also distinction between classical and quantum integrability, which briefly differentiate and describe the different methods for construction of invariants. The study of real, hermitian, complex and non-hermitian complex Hamiltonian systems are discussed briefly. We also consider nonlinear broken and unbroken $\mathcal{PT}$-symmetric Hamiltonian systems. Construction of invariants using rationalization method carried out in chapter 2. Since this method is being used for obtaining invariants for physical systems from a century ago, therefore we briefly describe the method for two dimensional TID system described by the Hamiltonian. We obtain invariants simple, shifted, coupled harmonic oscillator systems, non-hermitian Hamiltonians and also for Hamiltonian corresponding to KdV. We assume a general form of second constants of motion briefly known as invariant denoted by I up to second order. For all cases it is noticed that the invariant contains either even power or odd powers of momenta. We have computed for second order invariant in complex one and higher dimensions for oscillator system considering the crossed terms also.

In chapter 3, dynamic algebraic method has been used for construction of additional invariants corresponding to classical systems. This method is only applicable to TD systems, so we now focus our concentration on TD integrable systems, particularly on construction of second invariants in two dimensions. Complexification in two dimension is done by $\mathcal{PT}$-symmetry and the phase space used particularly known as ECPS (Extended Complex Phase Space). The coordinates of phase space in ECPS are $X = x_1 + ip_2; Y = x_2 + ip_3; P_x = p_1 + ix_3; P_y = p_2 + ix_4$. To the best of our knowledge, this ECPS approach to classical and quantum integrability is new in the literature and partially connects different approaches to integrability that are used in quantum physics.

In chapter 4, Quantum invariants are constructed for some systems in one and two-dimensions by adding appropriate quantum corrective terms to the corresponding classical invariants, which are constructed in chapter 3. For this purpose we have used Moyal's bracket and got some interesting result that under what condition a classical invariant becomes a quantum one. Since Classical integrable system (CI) is not necessarily Quantum Integrable (QI), therefore one has to study the quantum integrability aspects of nonlinear systems. Therefore the notion of quantum integrability is to be well understood. There is also a distinction between complete integrability, in the Liouville sense, and partial integrability, as well as a notion of superintegrability and maximal superintegrability. It seems natural to apply to the definition of classical integrability Dirac's prescription of replacing Poisson brackets by commutators of corresponding quantum operators $\{\ ,\ \} \to \frac{1}{\hbar}[\ ,\ ]$ in order to introduce the notion of quantum integrability. Finally the role and scope of some of the derived invariants in the context of various physical problems are highlighted. We highlight some applications of dynamical invariants in different branches of science which seems to be vital for understanding the system dynamics.

Chapter 1

Survey of Classical and Quantum Dynamical Systems

1.1 Introduction

Integrability of a dynamical system is a concept which is widely discussed for a long time in physics, mathematics and somewhat differently in different contexts. It can also be considered as a mathematical property that can be successfully used to obtain more predictive power and quantitative information to understand the system globally . A mathematical model may help to explain a system and to study the effects of different components, and to make predictions about behaviour. Integrability is one such property that can be successfully used [1, 2] to get more predictive power. There are various circumstances in which interest in integrability can arise.

In past non-linear ordinary and partial differential equations are generally used to study for the description of natural phenomena, such as weather changes, growth of population, non-linear dynamical systems, solitary waves, propagation of light in optical fibre, fluid and plasma turbulence, neuron's initiation and propagation, energy landscapes in glassy systems medical imaging. etc [3, 4, 5]. The evolution of typical dynamical systems is often described by nonlinear ordinary and partial differential equations. Characteristically, these nonlinear dynamical systems show regular as well as chaotic trajectories in phase space, depending on the number of dependent variables involved, the nature and the range of the external forces and the parameters involved, and the energy of the system. As a lot of parameters are involved in equation of motion, so sometime it difficult to find an exact analytical close form of solution of the problem. The assimilation, characterization, analysis and apportionment of regular and chaotic regimes of dynamical systems are in general obstructed by the absence of systematic and well defined analytical techniques to handle them.

It is, in fact, one of the important problems in nonlinear dynamics to identify when a given system displays regular motion. One has to identify the Hamiltonian or non-Hamiltonian systems, and should confirm that under what conditions the system becomes completely integrable and when it is non-integrable [6] exhibiting irregular or chaotic motion. Then naturally, the question which arises in this regard is: what is meant by integrability and how does it occur? The answer to the former question is somewhat vague as the concept of integrability is itself in a sense not well defined and there seems no unique definition for it yet. The latter is even more difficult to answer, as no well defined criteria seem to exist to identify integrable cases.

Integrability can be considered as a mathematical property that can be successfully used to obtain more prognosticative power and quantitative information to understand the dynamics of the system globally. Integrable models form a beautiful class of physical models in the sense that they are exactly solvable. For this reason they have always provided a treasured class of systems where the underlying structures of the model can be laid bare and understood. To study the cardinal and preeminent properties of a dynamical system, it is important to know that whether the system is integrable, and if it is, one wants to know as many quantities as possible whose values are conserved during the time evolution of the system [2, 6]. The global quantity mentioned above is a function form the space of dependent variables, of the phase space to the real (or may be complex) systems. This function is in the literature variously called as a integral of motion, conserved quantity, constant of motion, second invariant etc (since first one is Hamiltonian itself). There can be several, even an infinity of constants of motion depending on the number of degrees of freedom present in the system. One can learn a lot about nonlinear dynamical systems as the invariants are the analytic functions, and analytic results are much easier to use, to interpret and to generalize. Treating the integrable case as basic zeroth order exact solutions, it can also be further utilized to develop suitable schemes in order to deal with non integrable systems. From literature, one can follow that the integrability nature of a dynamical systems has been methodically investigated using the following two broad understandings [2].

The first one uses essentially the methodical meaning: integrable - integrated with the required number of integration constants; non-integrable - proven not to be integrable. This unconstrained definition of integrability can be related to the existence of single valued, analytic solutions, for non-linear ordinary or differential equations leading to the notion of integrability in the complex plane.

The second conception, particularly applicable to Hamiltonian systems, is to search for a sufficient number of single valued, analytic, involutive integrals for a Hamiltonian system with n-degrees of freedom, so that the associated Hamilton's equations of motion, in principle, can be integrated by quadratures in the sense of Liouville.

In the context of differentiable dynamical systems, the notion of integrability refers to the existence of invariant, regular foliations; i.e., ones whose leaves are embedded submanifolds of the smallest possible dimension that are invariant under the flow. There is thus a variable notion of the degree of integrability, depending on the dimension of the leaves of the invariant foliation. This concept has a refinement in the case of Hamiltonian systems, known as complete integrability in the sense of Liouville (see below), which is what is most frequently referred to in this context.

An extension of the notion of integrability is also applicable to discrete systems of lattice model such as Toda lattices. This definition can be adopted to describe evolution equations that either are finite difference equations or systems of differential equations.

The distinction between integrable and nonintegrable dynamical systems thus has the qualitative implication of regular motion vs. chaotic motion and hence is an intrinsic property, not just a matter of whether a system can be explicitly integrated in exact form.

1.2 Hamiltonian systems and Liouville integrability

We shall now move onto the next level in the formalism of classical mechanics, due initially to Hamilton around 1830. While we wont use Hamiltons approach to solve any further complicated problems, we will use it to reveal much more of the structure underlying classical dynamics. The basic idea of Hamiltons approach is to try and place position (x) and momentum (p) on a more symmetric footing.

In the special setting of Hamiltonian systems, we have the notion of integrability in the Liouville sense. Liouville integrability means that there exists a regular foliation of the phase space by invariant manifolds such that the Hamiltonian vector fields associated to the invariants of the foliation span the tangent distribution. Another way to state this is that there exists a maximal set of Poisson commuting invariants (i.e., functions on the phase space whose Poisson brackets with the Hamiltonian of the system, and with each other, vanish).

In finite dimensions, if the phase space is symplectic (i.e., the center of the Poisson algebra consists only of constants), then it must have even dimension $2n$, and the maximal number of independent Poisson commuting invariants (including the Hamiltonian itself) is n . The leaves of the foliation are totally isotropic with respect to the symplectic form and such a maximal isotropic foliation is called Lagrangian.

All autonomous Hamiltonian systems (i.e. those for which the Hamiltonian and Poisson brackets are not explicitly time dependent) have at least one invariant; namely, the Hamiltonian itself [7, 8, 9], whose value along the flow is the energy.

There are, of course, many approaches, which one can try to search invariants if the system seems to be integrable. However, in practical problems the prospects of integrability should be tested by other means, e.g. numerically and by singularity analysis [10, 11]. If the system does not fail either test, the next step would be to search for the invariant(s).

An integrable system in classical mechanics is one, which has n independent single valued first integrals of motion $F_m(q,p)$, here $m = 1, 2, ...n$ exist which are in involution and they satisfy

$$[F_m, F_n]_{PB} = \frac{\partial F_m}{\partial q_i}\frac{\partial F_n}{\partial p_i} - \frac{\partial F_m}{\partial p_i}\frac{\partial F_n}{\partial q_i},$$

where $[.,.]_{PB}$ is the Poisson bracket. If $F_n = H$ and the Poisson bracket vanishes then F_m is constant of motion. Classical Hamiltonian system of n degrees of freedom is said to be classical integrable if there are $(n-1)$ independent, well defined global function exist and there Poisson bracket with each other and with Hamiltonian $H(x, p, t)$vanishes, i.e.

$$\frac{dI}{dt} = [I, H]_{PB} = 0.$$

This implies that the constancy of I depends on the Hamiltonian, H, and in particular H itself is a constant of motion for time independent (TID) systems and so are the functions of H. For the invariant I to be nontrivial, we must, therefore, require that I is functionally independent of H. The global quantity mentioned above is a functional from the space dependent variables (phase space) to the real (or complex) numbers. This function in literature is called by different names such as "constant of motion","second invariant","conserved quantity" and "integral of motion" etc. There can be several constants of motion depending upon the number of degrees of freedom present in a system. The integral of motion is helpful in the reduction of the order of differential equation satisfied by a dynamical system. Once these invariants for a system are known then solution of equation of motion merely reduced to quadrature [11, 12].

Emmy Noether, 1918 (was an influential German lady mathematician) determined the conserved quantities for every system of physical laws that possesses some continuous symmetry. As an example, if a physical experiment has the same outcome at any place and at any time, then its laws are symmetric under continuous translations in space and time: by Noether's theorem, these symmetries account for the conservation laws of linear momentum and energy within this system, respectively.

First examples of a physical invariant is the speed of light under a Lorentz transformation and time under a Galilean transformation. Such space time transformations represent shifts between the reference frames of different observers, and so by Noether's theorem invariance under a transformation represents a fundamental conservation law. For example, invariance under translation leads to conservation of momentum, and invariance in time leads to conservation of energy. In the current era, the immobility of polaris (the North Star) under the diurnal motion of the celestial sphere is a classical illustration of physical invariance. The above discussion, clearly establishes the importance of invariants as far understanding of dynamics is concerned. In the next section, we give a brief description of various functional forms of invariants used in literature.

1.3 Different form of invariants

Invariants are important in modern theoretical physics, and many theories are expressed in terms of their symmetries and invariants. In our present work we are looking out for the second invariants, so it is essential to know what is meant by "invariants" of a dynamical system.

Consider a function I in phase space $F(x_i, p_i)$ for a Hamiltonian system. At the initial time t_0 the function I will get a value, $I(t_0)$, which is determined uniquely through the initial values $x_i(t_0)$ and $p_i(t_0)$, $i = 1, 2, ..., N$. As time evolves the coordinates evolve according to the Hamiltonian equations of motion and as a result, the value of I may also change. If, however, the value of I remains constant under the motion, then I is called a constant of motion or integral or invariant. Note that the constancy of I depends on the choice of the Hamiltonian H, and in particular H itself is a constant of motion for autonomous systems. As an invariant is basically a phase space function which is in involution with the Hamiltonian as well as other invariants, if they exist for the system.

For the invariant to be nontrivial, it is required that I must be functionally independent of H. Functional independency of two functions K and L is easily tested by considering $2N \times 2N$ Jacobian $\partial(K, L)/\partial(x_i, p_i)$ and if its rank is two then K and L are functionally independent. Although, various functional forms of invariants are investigated and discussed in literature [2], but here we describe only a few widely used forms of invariants.

Invariants in polynomial form

Energy of dynamical system usually expressed as sum of kinetic energy and potential energy. There is a special status of the kinetic energy term in the Hamiltonian. Therefore, a second invariant is often found [2, 6] to possess a polynomial form in momenta. Further the degree of polynomial is the order of the invariant.

In general, for a two dimensional system, an ansatz for a n'th order invariant can be made as

$$I = a_0 + a_i\xi_i + \frac{1}{2!}a_{ij}\xi_i\xi_j + \frac{1}{3!}a_{ijk}\xi_i\xi_j\xi_k + \frac{1}{4!}a_{ijkl}\xi_i\xi_j\xi_k\xi_l + ... \tag{1.1}$$

where $\xi_i = \dot{x}_i$ and $i, j, k, l, ... = 1, 2$. The coefficients $a_0, a_i, a_{ij}, a_{ijk}, a_{ijkl}, ...$ etc. are the functions of coordinates only and these coefficients are symmetric with respect to any interchange of their indices i.e. $a_{ij} = a_{ji}$ etc. Note that, for a autonomous Hamiltonian which is even power in momenta (i.e. it is invariant under the time reflection symmetry), the invariant will contain either even or odd power terms in momenta. However, if the potential term in H involves momentum-dependence, then all the terms upto desired order in the invariant should be considered.

Invariants in rational form

Now we consider invariants as rational [4, 13] functions of the momenta p's. In particular, invariant I for a system is considered of the form

$$I = \frac{R}{S},$$

where R and S are polynomials in momenta and $S \neq 0$, so that not only S^{-1} exists but also $RS^{-1} = S^{-1}R$ holds. Clearly, the vanishing of the Poisson bracket of this form of I and H requires

$$S[H, R]_{PB} = R[H, S]_{PB},$$

which is equivalent to the pair

$$[H,R]_{PB} = GR; \quad [H,S]_{PB} = GS,$$

with G as some rational function of momenta. It is worth to mention that if $I = R/S$ is an invariant of the system then

$$I = (aR + bS)/(cR + dS),$$

is also an invariant with a, b, c and d are arbitrary constants for which, $ad - bc \neq 0$.

Transcendental form of invariants

The generalization of rational form is called as transcendental form. Hence the rational invariant provides the transcendental invariants. By definition, a transcendental invariant [4, 13] I is an arbitrary function K of two different polynomials R and S in momenta

$$I = K(R,S).$$

The Poisson bracket of H and I yields

$$K_R[H,R] + K_S[H,S] = 0,$$

and the function K can be solved from the above equation only if

$$[H,R] = GP_1(R,S),$$

$$[S,H] = GP_2(R,S),$$

where P_i's are some polynomials in R and S. In this case, not only the functional form of K but also the degrees of polynomials R and S will suggest further classification of transcendental invariants.

In literature there also exist many other forms of invariants depending upon the requirement of a particular physical situation. In spite of large number of forms of invariants, the polynomial form is widely accepted one and utilized in different studies. We have also used the polynomial form of invariants in our studies. After a brief introduction of invariants, now we consider the notion of integrability in next the section.

1.4 Integrability

In physics and mathematics, completely integrable systems, especially in the infinite dimensional setting, are often referred to as exactly solvable models. This obscures the distinction between integrability in the Hamiltonian sense, and the more general classical dynamical systems sense. An imprecise notion of exact solvability as meaning: The existence of sufficient number of invariants, in terms of which the solutions may be expressed. Integrability of a dynamical system is a concept which is widely discussed for a long time in both physics and mathematics because it means the existence of some interesting geometrical structures in the phase space. In physical terms it is a property related with regular behavior and predictability. In all the areas of physics mathematical modeling which give rise to differential equations the modeling process includes the solution of those differential equations, be they (systems of) ordinary differential equations or partial differential equations. If this be possible in some sense, the system of differential equations is said to be integrable. (Note that this exclude numerical integration since this requires merely the existence of a continuous solution and that property can even be found in chaotic/turbulent systems.) A critical question is the meaning of Integrability. There are four possible ways to prescribe integrability. They are

(i) the ability to display a nonlocal functional equation involving the dependent and independent variables; this need not be explicit and, should the equation be implicit, the inversion by means of the Implicit function theorem need be no more than local,

(ii) the existence of a number of functionally independent first integrals/invariants equal to the order of the system in general and half that for a Lagrangian system as a consequence of Liouvilles theorem [1],

(iii) the existence of a sufficient number of Lie symmetries to reduce the differential equation (or system; unless otherwise obviously the singular implies the plural) to an algebraic equation and

(iv) the possession of the Painleve Property.

These above mentioned concepts are not entirely equivalent. In particular (iv) requires that the solution be analytic or possess no more than algebraic branch points in the complex plane (planes for more than one independent variable) and this is not demanded by (i), (ii) and (iii) although, of course, the idea that a solution must be analytic to be considered as a solution has been with us since the days of Poincare [5]. Even (i) and (ii) are not equivalent since it is not always possible to eliminate nonlocally the derivatives from the functionally independent first integrals/invariants. Case (iii) differs from (i) and (ii) since the final algebraic equation is in terms of the invariants of the symmetries used in the reduction of order and the reversal of the process on the assumption that a nonlocal solution of the algebraic equation exists requires a series of quadratures which one may not be able to perform in closed form. In the case of Lagrangian systems the celebrated theorem of Noether (references therein book [9, 12]) allows the identification of (ii) and (iii). The precise nature of the relationship between (iii) and (iv) has yet to be revealed. In the following two subsections we define the classical and quantum integrability.

1.4.1 Classical integrability

In classical mechanics the most common definition of integrability is that of Liouville integrability. Suppose we have a system with N degrees of freedom, that is, we have N coordinates q_i and N conjugate momenta p_j, with Poisson bracket $[q_i, p_j] = \delta_{ij}$. This system is said to be Liouville integrable if there are

N independent, well defined, global functions $I_k(p,q)$ such that

$$[I_i, I_j] = 0, \qquad \forall \;\; i, j,$$

i.e. all I_n's are involutive. In case of time independent (TID) systems, one of the I_n's is H itself. In principle, there can be more than N functionally independent invariants, but they may not be in involution.

List of some well-known classical integrable systems

1. Classical mechanical systems (finite-dimensional phase space):

- Harmonic oscillators in n dimensions
- Central force motion (Newton)
- Two center Newtonian gravitational motion (Euler)
- Geodesic motion on ellipsoids (Jacobi, 1838)
- Calogero-Moser-Sutherland models (in the 1970s)
- Neumann oscillator (lattices of N interacting particles, a variant of spherical harmonic oscillator)
- Swinging Atwood's machine with certain choices of parameters
- Integrable Clebsch and Steklov systems in fluids (Motion of a rigid body in ideal fluids in some special cases; Clebsch, Steklov, Kirchoff)
- Lagrange, Euler and Kovalevskaya tops (Motion of a rigid body about a fixed point.)

2. Integrable lattice models

- This type of Hamiltonian was first considered by Morikazu Toda. Toda lattice (solid state physics. It is given by a chain of particles with nearest neighbor interaction described by the equations of motion

$$\begin{aligned} \frac{d}{dt}p(n,t) &= e^{-(q(n,t)-q(n-1,t))} - e^{-(q(n+1,t)-q(n,t))}, \\ \frac{d}{dt}q(n,t) &= p(n,t), \end{aligned}$$

where q(n,t) is the displacement of the n-th particle from its equilibrium position, and p(n,t) is its momentum (mass m=1).)

The Hamiltonian of the Toda lattice is given by

$$H(q_1, ..., q_N, p_1, ..., p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}$$

- Ablowitz-Ladik lattice (nonlinear differential difference equations, Ablowitz-Ladik, 1975.)
- Volterra lattice $\{\dot{u}_i = u_i(u_{i+1} - u_{i-1}), i = 0, 1, 2, ...\}$

3. Integrable systems of PDEs in 1 + 1 dimension

- Korteweg-de Vries equation ($u_t = u_{xxx} + uu_x$; 1895)
- Sine-Gordon equation ($u_{tt} + u_{xx} = sinu$; 1898)
- Nonlinear Schrodinger equation ($i\partial_t\psi = -\frac{1}{2}\partial_x^2\psi + \kappa|\psi|^2\psi$)
- Boussinesq equation $\{\frac{\partial^2\omega}{\partial t^2} + \frac{\partial}{\partial x}(\omega\frac{\partial\omega}{\partial x}) + \frac{\partial^2\omega}{\partial^4 x}$; 1877$\}$
- Nonlinear sigma models $\{\mathcal{L} = \frac{1}{2}g(\partial^\mu\Sigma_a, \partial_\mu\Sigma_b) - V(\Sigma)\}$
- Classical Heisenberg ferromagnetic model (spin waves, Hamiltonian $\mathcal{H}$ for the Heisenberg ferromagnet: $\mathcal{H} = -\frac{1}{2}J\sum_{i,j}\mathbf{S}_i\cdot\mathbf{S}_j - g\mu_B\sum_i\mathbf{H}\cdot\mathbf{S}_i$, where J is the exchange energy, the operators S represent the spins at Bravais lattice points, g is the Land g-factor, μ_B is the Bohr magneton and $\mathbf{H}$ is the internal field which includes the external field plus any "molecular" field. Note that in the classical continuum case and in 1+1 dimensions Heisenberg ferromagnetic equation has the form $\mathbf{S}_t = \mathbf{S}\times\mathbf{S}_{xx}$.)
- Classical Gaudin spin system (Garnier system, 1919)
- Landau-Lifshitz equation (solid-state physics, the (LLE), is a PDE describing time evolution of magnetism in solids)
- Benjamin-Ono equation ($u_t + uu_x + Hu_{xx} = 0, 1967$, where H is the Hilbert transform.)
- Dym equation ($u_t = u^3u_{xxx}; 1966$, M. D. Kruskal.)
- Three wave equation

4. Integrable PDEs in 2 + 1 dimensions

- Kadomtsev-Petviashvili equation (1970, a generalization in 1+2 of the Boussinesq and KdV and equations)
- Davey-Stewartson equation ($iu_t + c_0u_{xx} + u_{yy} = c_1|u|^2u + c_2u\phi_x; 1974$, describe the evolution of a three-dimensional wave-packet on water of finite depth.)
- Ishimori equation (first example of a nonlinear spin-one field model in the plane that is integrable, Y. Ishimori, 1984)

5. Other integrable systems of PDEs in higher dimensions

- Self-dual Yang-Mills equations

There is also a distinction between complete integrability, in the Liouville sense, and partial integrability, as well as a notion of superintegrability and maximal superintegrability. Essentially, these distinctions correspond to the dimensions of the leaves of the foliation. When the number of independent Poisson commuting invariants is less than maximal (but, in the case of autonomous systems, more than one), we say the system is partially integrable. When there exist further functionally independent invariants, beyond the maximal number that can be Poisson commuting, and hence the dimension of the leaves of the invariant foliation is less than n, we say the system is superintegrable. If there is a regular foliation with one-dimensional leaves (curves), this is called maximally superintegrable.
A certain class of systems, possesses not only N but $2N-1$ independent constants of motion are known as superintegrable or overintegrable [14]. The best known cases of this particular class of systems are the N-dimensional harmonic oscillator, time dependent (TD) one dimensional harmonic oscillator.
List of some well-known superintegrable systems
1. Dynamical systems (finite-dimensional phase space):

- Harmonic oscillators in n dimensions (Energy, Angular momentum, Fradkin tensor).
- Central force motion (Kepler system; Energy, Angular momentum, Runge-Lenz vector).
- Hydrogen atom (quantum superintegrable systems)

Additional examples of superintegrable classical systems are the Fokas-Lagerstrom potential [15], the Smorodinsky-Winternitz potential [16], the Holt potential [17], and the Hartmann potentials [18] and the hydrogen atom in a linear electric field, the Toda chain [19]. In next section we will explain the quantum integrability of dynamical systems.

1.4.2 Quantum integrability

The apprehension of integrability in quantum mechanics comes naturally as it make a sweeping assumption of a similar type in classical mechanics. A quantum mechanical system described by the stationary Schrodinger equation

$$H\psi = E\psi, \qquad H = -\frac{1}{2}\nabla^2 + V(x_1, ..., x_n),$$

is completely integrable if there is a set of $(n-1)$ algebraically independent linear operators $X_a,\ a = 1, 2, ..., n-1$ commuting with the Hamiltonian and among each other

$$[H, X_a] = 0, \qquad [X_a, X_b] = 0.$$

The operators X_a, X_b are usually assumed to be polynomial in the momenta with coordinate dependent coefficients. The existence of such commuting operators lead to separation of variables in the Schrodinger equation. In the quantum setting, functions on phase space must be replaced by self-adjoint operators on a Hilbert space, and the notion of Poisson commuting functions replaced by commuting operators. Since there is no clear definition of independence of operators, except for special classes, the definition of integrable system, in the quantum sense, is not yet agreed upon. The working definition that is mostly used is that there is a maximal set of commuting operators, including the Hamiltonian, and a semiclassical limit in which these operators have symbols that are independent Poisson commuting functions on the phase space.

Quantum integrable systems can be explicitly solved by Bethe ansatz or quantum inverse scattering method. It has become widely known [21] that there is a one to one correspondence between operators $\hat{A}$ in Hilbert space and functions $A(q,p)$ on phase space

$$\hat{A} \rightleftharpoons A(q,p).$$

The above relationship is known as Wigner-Weyl transformation, which maps the quantum commutator $[\hat{A},\hat{B}]$ onto the Moyal bracket [22]

$$\begin{aligned} \{A,B\}_{MB} \quad &= \frac{2}{\hbar} A \sin((1/2)\hbar \overset{\leftrightarrow}{\wedge}) B \\ &= A \overset{\leftrightarrow}{\wedge} B - \frac{1}{24}\hbar^2 A \overset{\leftrightarrow}{\wedge}{}^3 B + \frac{1}{1920}\hbar^4 A \overset{\leftrightarrow}{\wedge}{}^5 B + ... \end{aligned} \tag{1.2}$$

Here $\{A,B\}_{MB}$ denotes the Moyal bracket and $\overset{\leftrightarrow}{\wedge}$ for a n dimensional system is given by

$$\overset{\leftrightarrow}{\wedge} = \sum_{i=1}^{n} \left[\frac{\overleftarrow{\partial}}{\partial q_i} \frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i} \frac{\overrightarrow{\partial}}{\partial q_i} \right]. \tag{1.3}$$

where $\partial^{(A)}$ acts only on the function A and $\partial^{(B)}$ on B. Therefore, in order to obtain the quantum invariant of the system from the corresponding classical one, the quantum corrections [21, 23] arising from the terms involving $\hbar$ in the expansion of the sine function in eq.(1.2) need to be incorporated. In the semiclassical limit $\hbar \to 0$ the Moyal bracket approaches the Poisson bracket

$$\{\hat{A},\hat{B}\}_{MB} \to A \wedge B = \{A,B\}_{PB}.$$

It is also mentioned that if the second invariant is at the most second order in momenta, then the Moyal bracket simply reduces to the Poisson bracket and the classical and quantum invariants turn out to be same. For cubic and higher order invariants, the need of quantum corrections arise and only after these corrections a classical invariant becomes quantum invariant in the spirit of Moyal bracket. Constants of motion for quantum systems are obtained by adding quantum correction terms, computed using Moyals bracket, to the corresponding classical counterparts.

List of some well-known Quantum integrable systems

- Lieb-Liniger model

- Hubbard model
- Heisenberg quantum model

In order to find integrable systems, one needs to find sufficient number of invariants of a system. Therefore various methods are being devised to meet the quest of invariants and in the following section we discuss some important systems and methods which have been distinctly proved their usefulness.

Generalized Ermakov systems

Ermakov systems attracted the attention of a great number of research people (see ref. [1] and references therein) since the last decades. Its central idea was first pointed out by Ermakov (as cited by Noether in his article [35]) about a century ago i.e. 1885, but the main push on the subject found motivation in two special virtues of this class of coupled nonlinear systems of equations: velocity between the two invariants and provides:
(a) Both simple and generalized Ermakov systems possess general solutions that simplify the nonlinear superposition law
(b) Errnakov systems are amenable to straightforward quantization. In this respect they provide nice examples of direct quantization of nonautonomous systems.
In 1885 Ermakov obtained a first integral for the time-dependent harmonic oscillator,

$$\ddot{x} + \omega^2(t)x = 0.$$

by introducing an auxiliary equation

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3}.$$

Integral is obtained by eliminating the w^2 between above two equations and multiplying by the integrating factor $(\rho\dot{x} - x\dot{\rho})$. It is

$$I = \frac{1}{2}[(\rho\dot{x} - x\dot{\rho})^2 + (x/y)^2] \tag{1.4}$$

which is usually called the Lewis invariant, following Lewis rediscovery of it in 1966 using the asymptotic method of Kruskal [32].
The Ermakov system as discussed by Ray [19], Reid [65] and Lutzky [25] among others. These authors consider the pair of equations

$$\ddot{x} + \omega^2(t)x = \frac{1}{x^2 y} f(y/x)$$

and

$$\ddot{y} + \omega^2(t)y = \frac{1}{y^2 x} g(x/y)$$

are associated with invariant

$$I = \frac{1}{2}[(\dot{x}y - \dot{y}x)^2 + V(r)]$$

where $V(r) = \int_0^r g(z)dz + 2\int_0^{\frac{1}{r}} f(z)dz$ and they obtained the third invariant using nonlinear superposition law [2] known as the Ermakov-Ray-Reid invariant

$$I = \frac{1}{2}(\dot{x}y - \dot{y}x)^2 + 2\int_0^{\frac{y}{x}} f(z)dz + 2\int_0^{\frac{y}{x}} dz \int_0^z \frac{1}{u}g(u)du - \frac{2Ky}{x}$$

f and g depend arbitrarily on their argument and K is a constant.

Different methods have been developed to build invariants of mechanical and physical systems, such as the Ermakov technique [4, 5], the symmetries approach [8] and the discrete symmetries approach [11], the dynamical algebraic method [2, 21, 39] and the algebraic structure and Poisson methods [37, 39]. Of all the methods, those making use of the algebraic structure present the additional advantage of being extended in a straightforward way to the corresponding quantum mechanical systems. In recent year, the use of a Lie algebraic approach [12] to build dynamical invariants has provided many interesting results. Several derivations of the dynamical invariant have been proposed in the literature: The exact invariant (1.4) was first derived by Lewis [1,2] from Kruskals adiabatic invariant. Leach [33] used time-dependent canonical transformations, Lutzky [25] derived the invariant (1.4) from Noethers theorem and recently Ray and Reid [24] presented a direct pro the dynamical invariance of (1.4), which assumes, however, that the auxiliary equation is already known. A physical interpretation of the dynamical algebra generated by the Hamiltonian the invariant has been given by Eliezer and Gray [34] in terms of the angular momentum of an auxiliary rotational motion.

1.5 Different methods for search of invariants

The Hamiltonian dynamics systems have been played an important role not only in modern physics, but also in mathematics, mechanics, engineering science, and social sciences, especially in nonlinear science, celestial mechanics, and spacecraft attitude dynamics [16]. But traditional Hamiltonian systems theory is defined in even dimensionality space where good characters have on the structure, so also limit its application. In 1959, the physicist, Martin D. Kruskal [32] was trying to promote the methods of Hamiltonian to apply it to the system that does not have existence of Lagrangian, and obtained the same promotion. Their results, as a promotion theory of the Hamiltonian system, was called generalized Hamiltonian mechanical systems. Since then, the study of the generalized Hamiltonian systems have became a hot topic, and they have widely develop the theories and applications, and this included a series of results. The principle of symmetry is a higher level of law in physics, and conserved quantities of the dynamical systems can better reveal the profound physical laws. The conserved quantity of a physical system has a close relation with its symmetry. In 1918, a German woman scientist, A. E.

Noether [35] first discovered that conserved quantities correspond to some symmetries, then conserved quantities can be found through symmetries of the system, and since then the theory of Noether symmetry was established. In 1979, M. Lutzky [25] applied Lies theory to the differential equations of motion for mechanical systems, and studied Lie symmetries and conserved quantities of a dynamical system. In 1992, Hojman [36] gave a new conservation theorem, and the conserved quantity was constructed in terms of a symmetry transformation vector of the equations of motion only. In the past 20 years, researchers developed the methods of Noether symmetry and Lie symmetry, and obtained many important results [23-25].

There have been many mathematical methods developed in the past [2, 6, 10, 11, 25, 26, 27, 29, 30, 38, 43] to obtain involutive second constants of motion, that span from elementary algebraic methods to symmetry considerations evaluated through symplectic group transformations or Noether's theorem. Recently researcherseveral author have applied some new methods for construction of invariant [31, 32, 38, 43]. But none of these methods have a universal character, and in most of the cases one or more adhoc assumptions are to be made for obtaining concrete results. Here, however, we describe only a few methods for construction of invariants of desired order.

1. Rationalization method

Whittaker [1] introduced the rationalization method for the construction of invariants, second order in momenta, of TID systems. Subsequently, this method has been used by several workers for finding invariants of both TID and TD system in one and two dimensions [6, 10, 30, 31, 35, 36, 39, 41]. Since this method is being used for obtaining invariants systems, therefore, we briefly describe the method considering a two dimensional TID dynamical system described by the Hamiltonian as

$$H = \frac{1}{2}(p_x^2 + p_y^2) - V(x, y), \tag{1.5}$$

And simultaneously for three dimensional TID dynamical system described by the Hamiltonian as

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - V(x, y, z), \tag{1.6}$$

Assuming the existence of a second constant of motion, I (fourth order polynomial in momenta) for the system, eq.(1.4), as

$$I = a_0 + a_i\xi_i + \frac{1}{2!}a_{ij}\xi_i\xi_j + \frac{1}{3!}a_{ijk}\xi_i\xi_j\xi_k + \frac{1}{4!}a_{ijkl}\xi_i\xi_j\xi_k\xi_l + ... \tag{1.7}$$

As it is noticed earlier that for TID systems, the invariant contains either even powers or odd powers of momenta [33]. Hence the above equation reduces to

$$I = a_0 + \frac{1}{2!}a_{ij}\xi_i\xi_j + \frac{1}{4!}a_{ijkl}\xi_i\xi_j\xi_k\xi_l, \tag{1.8}$$

where $i, j, k, l, ... = 1, 2, 3$, $\xi_1 = \dot{x}_1$, $\xi_2 = \dot{x}_2$, $\xi_3 = \dot{x}_3$ and $a_0, a_i, a_{ij}, a_{ijk}, a_{ijkl}$, etc. are functions of x_1, x_2, x_3 only.

The invariance of the function I implies $dI/dt = 0$ i.e. is Poisson bracket and H is the Hamiltonian of the system. On rationalizing the expression, obtained after using eq.(1.7)

$$\frac{dI}{dt} = [I, H]_{PB} = 0, \tag{1.9}$$

where $[..]_{PB}$ in eq.(1.9), with respect to the power of ξ_i, ξ_j, ξ_k... etc. and their all possible products, we get a system of over-determined coupled first order differential equations for the coefficient functions $a_0, a_i, a_{ij}, a_{ijk}, a_{ijkl}, ...$ etc. The mutually consistent solutions of these partial differential equations (PDEs) for potential V, called here the "potential" equations gives the invariant. As this method gives exact invariants for a system, one can utilize it to find higher order invariants for both real and complex Hamiltonian systems in two or higher dimensions. We will elaborate the rationalization method in chapter 2 and 4 for construction of higher order classical and quantum invariants of a number of systems.

2. Lie-Algebraic approach

For obtaining invariants of a variety of TD systems, researchers [2, 21, 37, 39] used the closure property of dynamical Lie-algebra generated by phase space functions. In this approach one can express the Hamiltonian $H(x, y, p_x, p_y, t)$ of the system as

$$H = \sum_n h_n(t)\Gamma_n(p_x, p_y, x, y, t), \tag{1.10}$$

where the set of functions $\{\Gamma_1, ..., \Gamma_n\}$ are not explicitly TD and coefficient $h_n(t)$'s are functions of time. The Γ_n's in eq.(1.10) generate a closed dynamical algebra, implies

$$[\Gamma_n, \Gamma_m] = \sum_l C^l_{nm}\Gamma_l, \tag{1.11}$$

where C^l_{nm} are the structure constants of the algebra. If the Γ_n's in eq.(1.11) are not sufficient to close the algebra then the set of Γ_n must be extended by adding new Γ_l's, such that $\Gamma_l = [\Gamma_n, \Gamma_m]$, until the closure is obtained along with additional $h_l(t)$'s which are taken to be zero. It is worth to mention that the algebra contains the important structural information for the dynamical behaviour of the system besides its straightforward extension to the quantum realms. Since the dynamical invariant I is also a part of Lie-algebra, then one can express this as

$$I(t) = \sum_k \lambda_k(t)\Gamma_k(p_x, p_y, x, y), \tag{1.12}$$

where $\lambda_k(t)$'s are TD coefficients. The invariance of $I(t)$ for a TD system requires

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H]_{PB} = 0. \tag{1.13}$$

Thus, using eq.(1.10) and eq.(1.12) for H and I respectively in eq.(1.13), we get a system of linear, first order differential equations, namely

$$\dot{\lambda}_r + \sum_n \left[\sum_m C^r_{nm} h_m(t)\right]\lambda_n = 0, \tag{1.14}$$

in λ_n's. Therefore, the solutions of these differential equations in turn provide classical invariant of a given system.

Since Lie-algebraic method gives exact TD invariants for classical systems and can easily be extended in quantum domains, therefore, we have elaborated this method while constructing complex invariants in 3rd chapter.

3. Transformation-Group method

J. Ray [19] and Burgan *et al* [20] has introduced and developed this method, based upon the transformation-group techniques, which deals with the transformation of both dependent and independent variables. The unknown coefficient functions of the transformation are set in such a way that the form of the equation of motion remains invariant under the transformation. Interestingly, the energy-integral in the new coordinates turns out to be the desired invariant of the system.

Here, we demonstrate the method for one dimensional TDHO whose equation of motion is given as

$$\ddot{x} + \omega^2(t)x = g(t). \tag{1.15}$$

For the system, eq.(1.15), we use the transformations

$$x' = \frac{x}{C(t)} + A(t); \qquad t' = D(t), \tag{1.16}$$

where C, A and D are arbitrary functions of time t. Under these transformations, eq.(1.15) takes the form

$$\begin{aligned} &C\dot{D}^2\frac{d^2x'}{dt'^2} + (2\dot{C}\dot{D} + C\ddot{D})\frac{dx'}{dt'} + [\ddot{C} + \omega^2(t)C]x' \\ &+[-\ddot{C}A - 2\dot{C}A - \omega^2(t)CA - C\ddot{A} - g] = 0. \end{aligned} \tag{1.17}$$

Demanding that the form eq.(1.9) remains invariant under eq.(1.16), the coefficients of (dx'/dt') in eq.(1.17) must vanish. This yields

$$\dot{D} = dt'/dt = 1/C^2,$$

and, hence eq.(1.15) becomes

$$\frac{d^2x'}{dt'^2} + C^3[\ddot{C} + \omega^2(t)C]x' + C^3[-\ddot{C}A - 2\dot{C}A - \omega^2(t)CA - C\ddot{A} - g] = 0. \tag{1.18}$$

In order to identify eq.(1.18) with the equation (i.e. with the equation of motion for a TID HO)

$$\frac{d^2x'}{dt'^2} + kx' = 0, \tag{1.19}$$

one has to choose A and C in eq.(1.18) such that

$$\ddot{C} + \omega^2(t)C = k/C^3, \tag{1.20}$$

$$\ddot{A} + (kA/C^4) + (2\dot{C}\dot{A}/C) + g/C = 0. \tag{1.21}$$

The energy integral for eq.(1.19) has the form

$$I = \frac{1}{2}\left(\frac{dx'}{dt'}\right)^2 + kx'^2, \tag{1.22}$$

and after carrying out the inverse transformation, the integral turns

$$I = \frac{1}{2}(C\dot{x} - \dot{C}x + C^2\dot{A})^2 + \frac{1}{2}(x/C + A)^2. \tag{1.23}$$

Here, C and A are the solutions of eqs.(1.20) and (1.21) respectively. Leach [33] has also used these transformations to find the invariants for some autonomous systems. This method has been widely used by used to obtain an exact solution of TD Schrodinger equation for three dimensional nonlinear potentials.

4. Lutzky's approach

This method, based on the formulation of Noether's theorem [35], is due to Lutzky [25]. This method is also used by many authors [4, 29, 32] for TD systems in one dimensional systems. For a TD system, the symmetry transformation is described by a group operator

$$X = \xi(x,t)\frac{\partial}{\partial t} + \eta(x,t)\frac{\partial}{\partial x}. \tag{1.24}$$

If the symmetry transformations defined by eq.(1.24) leaves the action

$$A = \int L(x,\dot{x},t)dt, \tag{1.25}$$

invariant, then the combination of the terms $\xi\frac{\partial L}{\partial t} + \eta\frac{\partial L}{\partial x} + (\dot{\eta} - \dot{x}\dot{\xi})\frac{\partial L}{\partial \dot{x}} + \dot{\xi}L$, is a total time derivative of the function $f(x,t)$, i.e.

$$\xi\frac{\partial L}{\partial t} + \eta\frac{\partial L}{\partial x} + (\dot{\eta} - \dot{x}\dot{\xi})\frac{\partial L}{\partial \dot{x}} + \dot{\xi}L = \dot{f}. \tag{1.26}$$

It follows from this that a constant of motion for the system is

$$I = (\dot{x}\xi - \eta)\frac{\partial L}{\partial \dot{x}} - \xi L + f. \tag{1.27}$$

In eq.(1.26) $\dot{\eta}, \dot{\xi}$ and $\dot{f}$ are defined as

$$\dot{\xi} = \frac{\partial \xi}{\partial t} + \dot{x}\frac{\partial \xi}{\partial x}; \quad \dot{\eta} = \frac{\partial \eta}{\partial t} + \dot{x}\frac{\partial \eta}{\partial x}; \quad \dot{f} = \frac{\partial f}{\partial t} + \dot{x} \tag{1.28}$$

This method is successfully applied not only to TD harmonic oscillator (TDHO) but also to several of its generalizations.

5. Method by Struckmeier and Riedel

Jürgen Struckmeier and Claus Riedel [38] have formulated a method for construction of exact invariants for TD classical Hamiltonians systems. Consider a system of a nonrelativistic ensemble of N particles of the same species moving in an explicitly time-dependent and velocity-independent potential, whose Hamiltonian H takes the form

$$H = \sum \frac{1}{2}[P_x^2 + P_y^2 + P_z^2] + V(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}, \overrightarrow{t}) \tag{1.29}$$

with $\overrightarrow{x}, \overrightarrow{y}, and \overrightarrow{z}$ the N component vectors of the spatial coordinates of all particles. It is hereby assumed that the system may be completely described within $6N$-dimensional Cartesian phase space spanned by the $3N$ particle coordinates and their conjugate momenta. From the canonical equations, we derive for each particle i the equations of motion

$$\dot{x} = p_x \; ; \qquad \dot{p}_x \;=\; -\frac{\partial V(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{x}, t)}{\partial x}. \tag{1.30}$$

and likewise for the y and z degrees of freedom. The solution functions $\overrightarrow{x}(t), \overrightarrow{y}(t), \overrightarrow{z}(t)$ and $\overrightarrow{p_x}(t), \overrightarrow{p_y}(t), \overrightarrow{p_z}(t)$ define a path within the $6N$-dimensional phase space that completely describes the system's time evolution. A quantity

$$I = I[\overrightarrow{x}(t), \overrightarrow{y}(t), \overrightarrow{z}(t),\ \overrightarrow{p_x}(t),\ \overrightarrow{p_y}(t),\ \overrightarrow{p_z}(t)] \tag{1.31}$$

constitutes an invariant of the particle motion if its total time derivative vanishes:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \sum \left[\frac{\partial I}{\partial x}\dot{x} + \frac{\partial I}{\partial y}\dot{y} + \frac{\partial I}{\partial z}\dot{z} + \frac{\partial I}{\partial p_x}\dot{p}_x + \frac{\partial I}{\partial p_y}\dot{p}_y + \frac{\partial I}{\partial p_z}\dot{p}_z \right] = 0 \tag{1.32}$$

We examine the existence of a conserved quantity (1.31) for a system described by (1.29) with a special ansatz for I being at most quadratic in the velocities

$$I = \left[f_2(t) + (P_x^2 + P_y^2 + P_z^2) + f_1(x,t)p_x + g_1(x,t)p_y + h_1(x,t)p_z \right] + f_0(\overrightarrow{x}(t), \overrightarrow{y}(t), \overrightarrow{z}(t), t) \tag{1.33}$$

The set of functions $f_2(t), f_1(x,t), g_1(x,t), h_1(x,t)$ and $f_0(\overrightarrow{x}t, \overrightarrow{y}t, \overrightarrow{z}t, t)$ that render I invariant are to be determined.

6. Lax-Pair method

This method for solving the nonlinear differential equations is due to P. D. Lax [26]. In order to obtain the working knowledge of Lax-Pair method, consider

$$\frac{d\mathbf{u}}{dt} = V(u), \qquad \mathbf{u} \equiv (u_1, u_2, ..., u_m),$$

be an autonomous system of first order ODEs. Assume that this system can be written as

$$\frac{dL}{dt} = [A, L](t) \equiv [A(t), L(t)], \tag{1.34}$$

where A and L are $n \times n$ matrices and $[A, L] \equiv AL - LA$. The $n \times n$ matrices L and A are called a Lax pair and the eq.(1.34) is known as Lax representation.

For finding integrals of motion one can write eq.(1.34) as

$$\frac{dL^k}{dt} = [A, L^k](t),$$

then $tr(L^k)$ $(k = 1, 2, ...)$ are integrals of motion and if L^{-1} exists, then $tr(L^{-1})$ is also an invariant, where tr(.) denotes the trace. Here we consider an example to elaborate the method outlined above.

Let H be the Hamiltonian function given as

$$H(x,p) = \frac{1}{2}(p_1^2 + p_2^2) + e^{x_2 - x_1}. \tag{1.35}$$

Now define

$$
\begin{aligned}
a &= \frac{1}{2}e^{(x_2-x_1)/2},\\
b &= \frac{1}{2}p_1,\\
c &= \frac{1}{2}p_2.
\end{aligned}
\tag{1.36}
$$

The Lax matrices for the system is identified as

$$
L = \begin{pmatrix} b & a \\ a & c \end{pmatrix}; \quad A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}. \tag{1.37}
$$

Then the Lax representation using eq.(1.37) is written as

$$
[A, L] \equiv AL - LA = \begin{pmatrix} 2a^2 & a(c-b) \\ a(c-b) & -2a^2 \end{pmatrix}. \tag{1.38}
$$

The integrals of motion are given by

$$
I_1(a, b, c) = tr(L) = b + c,
$$

$$
I_2(a, b, c) = tr(L^2) = 2a^2 + b^2 + c^2.
$$

One can find invariants of other systems by identifying Lax pair L and A. Consider two operators $L(x, y, t)$ and $A(x, y, t)$, and satisfying the operator equation $[L, A] = -\frac{\partial L}{\partial t}$, the eigen value λ of L such that $L\psi = \lambda\psi$, is independent of t if and only if the corresponding eigenfunction ψ evolves in time t according to $A\psi = -\frac{\partial \psi}{\partial t}$. Note that if $[L, A] = -\frac{\partial L}{\partial t}$ is equivalent to the original Hamilton's equations of motion, then the terms in the expansion of $\det(L - H)$ involve invariants and the latter are in involution.

7. Field method approach (oscillators with odd potentials)

In this Approach, Ivana Kovacic *et al* [43] has obtained some approximate or adiabatic invariants of non-linear autonomous oscillators:

$$
\ddot{x} + G(x) = 0; \tag{1.39}
$$

$$
x(0) = a; \quad \dot{x}(0) = 0.
$$

where $G(x)$ is an odd function of a coordinate x, which does not necessarily have a linear term and overdots denote differentiation with respect to time t. The problem is approached by the field method technique [44], which has been approved as beneficial for studying different problems of disparate areas of mechanics.

In order to apply the field method algorithm developed for obtaining conservation laws of the linear one-degree of freedom oscillators, the system (1.39) can be written down as:

$$
\dot{x} = p\;; \quad \dot{p} = -\omega^2 x + F, \tag{1.40}
$$

where

$$F \equiv F(t) + \omega^2(x) + G(x(t)). \tag{1.41}$$

and ω is the frequency to be found. Then, the basic assumption of the field method will be introduced, which is that the coordinate x can be represented as a field depending on time t and the momentum (i.e. the velocity) p

$$x = U(t, p)$$

Partial differentiation of the above expression in combination with (1.40) yields

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial p} + [-\omega^2 U + F(t)] - p = 0. \tag{1.42}$$

The solution of this partial differential equation can be assumed in the form [45]:

$$U = Ap + f(p)$$

with A being a constant and f and t being an unknown function of time. Substituting it into (1.42) and equating the terms involving p and the free terms with zero, one has:

$$A_{\frac{1}{2}} = \sqrt{A} = \pm\frac{i}{\omega}, \quad f(t) = Ce^{A\omega^2 t} - Ae^{A\omega^2 t}\int F(\tau)e^{A\omega^2\tau}d\tau. \tag{1.43}$$

where i is an imaginary unit and C is a constant. For two values of the constant A (1.43), algebraic transformations of the assumed form ($U = Ap + f(p)$) lead to the expressions in which the convolution integrals [46] appear:

$$x - \frac{ip}{w} + \frac{i}{w}\int[\omega^2(\tau) - G(x(\tau))]e^{-\omega(\tau-t)}d\tau = C_1 e^{i\omega t} \tag{1.44}$$

$$x - \frac{ip}{w} - \frac{i}{w}\int[\omega^2(\tau) - G(x(\tau))]e^{-\omega(\tau-t)}d\tau = C_1 e^{i\omega t} \tag{1.45}$$

To solve the integrals in (1.44) and (1.45), the solution for the coordinate x inside the square brackets is assumed as $x(\tau) \approx \frac{C_1^{i\omega\tau} + C_2 e^{-i\omega\tau}}{2}$ In accordance with the initial conditions (1.39), this form gives the constants C_1 and C_2:

$$C_1 = C_2 = a.$$

Now, the expressions (1.44) and (1.45) can be presented as:

$$\left[x - \frac{ip}{\omega}\right]e^{-i\omega t} + \frac{i}{\omega}\int\left[\omega^2\frac{a + ae^{-i2\omega\tau}}{2} - e^{-i\omega t}G\left(\frac{ae^{i\omega\tau} + ae^{-i\omega\tau}}{2}\right)\right]d\tau = a.$$

$$\left[x + \frac{ip}{\omega}\right]e^{-i\omega t} - \frac{i}{\omega}\int\left[\omega^2\frac{a + ae^{i2\omega\tau}}{2} - e^{i\omega t}G\left(\frac{ae^{i\omega\tau} + ae^{-i\omega\tau}}{2}\right)\right]d\tau = a. \tag{1.46}$$

the frequency x will be calculated from the elimination of secular terms among the terms generated by the integrals. Eliminating the secular terms and integrating the remaining terms, some function of time will be obtained. Together with the terms in front of the integrals, they will form two independent linear adiabatic invariants. They provide additional information about the behavior of the system being considered, giving us the combinations of the parameters of the system which remain almost constant during time. Besides, they enable us to find a quadratic approximate invariant as their product. It can be presented in the form:

$$I = x^2 + \frac{p^2}{\omega^2} + xD_1(t) + pD_2(t) + D_3(t) - a^2. \tag{1.47}$$

where $D_1(t), D_2(t)$ and $D_3(t)$ stand for some functions of time. Depending explicitly on time, the quadratic form (1.47) differs from the corresponding exact invariant (total energy conservation law) of the system.

8. Prelle-Singer method

Prelle-Singer (PS) [27] proposed a procedure for solving first order ODEs in terms of elementary functions. The attractiveness of PS-method is that if the given system of first order ODEs has a solution in terms of elementary functions then the method guarantees that this solution will be found. Recently, authors [47, 48] used this method for second order ODEs and obtained first integrals. Now we present a brief description of the PS-method for obtaining invariants, which follows as :

Consider a second order ODE as

$$\ddot{x} = \frac{P}{Q}, \qquad P, Q \in \ C[t, x, \dot{x}], \tag{1.48}$$

where over dot denotes differentiation with respect to time and P and Q are polynomials in t, x and $\dot{x}$ with coefficients in the field of complex numbers. Let us assume that the ODE, eq.(1.48), admits a first integral $I(t, x, \dot{x}) = C$, with C constant on the solutions, so that the total differential gives

$$dI = I_t dt + I_x dx + I_{\dot{x}} d\dot{x} = 0, \tag{1.49}$$

where the subscripts denote partial differentiation with respect to the variables t, x and $\dot{x}$. Rewriting eq.(1.49) in the form $\frac{P}{Q}dt - d\dot{x} = 0$ and adding a null term $S(t, x, \dot{x})\dot{x}dt - S(t, x, \dot{x})dx$ to the latter, we obtain that on the solutions the I-form

$$\left(\frac{P}{Q} + S\dot{x}\right)dt - Sdx - d\dot{x} = 0. \tag{1.50}$$

Hence, on the solutions, the I-forms, eq.(1.49) and eq.(1.50), must be proportional. Multiplying eq.(1.49) by the factor $R(t, x, \dot{x})$ which acts as the integrating factor for eq.(1.50), we have on the solutions that

$$dI = R(\phi + S\dot{x})dt - RSdx - Rd\dot{x} = 0, \tag{1.51}$$

where $\phi \equiv \frac{P}{Q}$. Comparing eq.(1.49) with eq.(1.51) we have, on the solutions, the relations

$$
\begin{aligned}
I_t &= R(\phi + S\dot{x}), \\
I_x &= -RS, \\
I_{\dot{x}} &= -R.
\end{aligned} \tag{1.52}
$$

Therefore, the compatibility conditions, $I_{tx} = I_{xt}, I_{t\dot{x}} = I_{\dot{x}t}, I_{x\dot{x}} = I_{\dot{x}x}$, between the eqs.(1.52), require that

$$D[S] = -\phi_x + S\phi_{\dot{x}} + S^2, \tag{1.53}$$

$$D[R] = -R(S + \phi_{\dot{x}}), \tag{1.54}$$

$$R_x = R_{\dot{x}} + RS_{\dot{x}}, \tag{1.55}$$

where

$$D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial \dot{x}}.$$

Eqn.(1.53)-(1.55) can be solved in the following way. Substitute the given expression of ϕ into eq.(1.53) and solve it for S. Once S is known then eq.(1.54) gives the value of R. Now the functions R and S have to satisfy an extra constraint, that is, eq.(1.55). Once a compatible solution satisfying all the three equations have been found, then the functions R and S fix the integral of motion $I(t, x, \dot{x})$ by the relation

$$
\begin{aligned}
I(t, x, \dot{x}) = \int R(\phi + \dot{x}S)dt - \int (RS + \frac{d}{dx}\int R(\phi + \dot{x}S)dt)dx - \\
\int \left[R + \frac{d}{d\dot{x}}[\int R(\phi + \dot{x}S)dt - \int (RS + \frac{d}{dx}\int R(\phi + \dot{x}S)dt)dx]\right]d\dot{x}
\end{aligned} \tag{1.56}
$$

Hence for every independent set (S, R), eq.(1.56) defines an integral. We shall again return to this method, in chapter 6, for finding solutions of second order ODEs if the invariants of the system are known.

9. Darboux integrability method

In 1878 Darboux [28] initiated the theory of planar polynomial differential systems, and his work provided a link between algebraic geometry and the search of first integrals. He demonstrated how to construct first integrals of polynomial vector fields. Darboux integrability that lies at the very heart of the notions of Liouville integrability and the Prelle-Singer theorem [27], which are essentially built on its foundation. It has to be mentioned that the classical and powerful method of symmetry analysis, as formulated by Lie, is an important tool for finding solutions of differential equations and includes various methods for determining first integrals. Here we describe the modified Darboux theory of integrability for polynomial ODEs in three and more dimensions. Darboux method of integrability is one of the best known methods for finding first integrals of polynomial ODEs.

Consider a system of two first-order ODEs of the form

$$\begin{aligned}\frac{dx_1}{dt} &= X_1(t, x_1, x_2),\\ \frac{dx_2}{dt} &= X_2(t, x_1, x_2)\end{aligned} \tag{1.57}$$

A solution of above, namely, $x_1 = x_1(t)$, $x_2 = x_2(t)$, assuming the values $x_1(0), x_2(0)$ at $t = t_0$ say, defines in space a certain curve, which passes through the point $P_0(t_0, x_1(0), x_2(0))$, and is called an integral curve of the system.

Let us consider planar polynomial differential systems,

$$\dot{x} = Q(x, y) \qquad and \qquad \dot{y} = P(x, y),$$

where $P(x, y) = \sum_i = 0^m P_i(x, y)$, $Q(x, y) = \sum_i = 0^m Q_i(x, y)$ are co-prime polynomials in $\mathcal{C}$ such that max{deg P,deg Q}=m and $P_i(x, y)$ and $Q_i(x, y)$ are homogeneous components of degree i. The planar differential system (1.57) may alternatively be described by the following vector field:

$$D = Q(x, y)\frac{\partial}{\partial x} + P(x, y)\frac{\partial}{\partial y} \tag{1.58}$$

or a differential form $\omega = Pdx - Qdy$, The corresponding phase-flow being given by the solutions of first-order ODE,

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}. \tag{1.59}$$

The tangents to the trajectories of a planar polynomial differential system are defined everywhere. If $f(x, y) = 0$ is the equation of an invariant curve, its tangent must coincide with the tangents of the trajectories. In other words, the gradient to $f, \nabla f = (\frac{\partial f}{\partial x}\frac{\partial f}{\partial y})$, and (Q, P) must be orthogonal over the curve $f(x, y) = 0$,

$$\dot{f} = \left(Q\frac{\partial f}{\partial x} + P\frac{\partial f}{\partial y}\right)_{f=0} = 0. \tag{1.60}$$

An invariant curve $f(x, y) = 0$ is called an algebraic curve of degree m when $f(x, y)$ is a polynomial of degree m. Let D be the vector field associated with differential equation. A curve $f(x, y) = 0$ is an *invariant* algebraic curve if $D[f]/f$ is a polynomial. The latter polynomial $\lambda_f = D[f]/f$ is usually called the cofactor of the invariant algebraic curve.

10. Painlevé method

The basic idea of Painlevé (P) analysis is to identify and characterize the nature of the singularities admitted by the general solution of ordinary differential equations (ODEs) and PDEs in the complex plane [7, 8, 9] of the independent variable. Therefore, for an ODE to be P-type, it is necessary that it has no movable branch points, either algebraic or logarithmic. For the study of Painlevé method, we describe

Ablowitz, Ramani and Segur (ARS) conjecture[7], which provides a systematic way to investigate the presence of movable branch-points and to determine whether the given ODE is P-type or not.

Let us consider a nth order ODE,

$$\frac{d^n w}{dz^n} = F(z; w, dw/dz, ..., d^{n-1}w/dz^{n-1}), \tag{1.61}$$

or equivalently, n first-order equations as

$$\frac{dw_i}{dz} = F_i(z; w_1, w_2, ..., w_n), \qquad i = 1, 2, ..., n, \tag{1.62}$$

where F and F_i are analytic in z and rational in their arguments. Then the ARS conjecture essentially consists of the following three steps.

(a) Determination of leading-order behaviour of the Laurent series in the neighborhood of the movable singular point z_0;

(b) Determination of resonances, that is, the power at which arbitrary constants of the solution of eq.(1.61) can enter into the Laurent series expansions; and

(c) Verification that a sufficient number of arbitrary constants exist without the introduction of movable critical points.

At the end of the three steps one will be in position to check the necessary conditions for the existence of the P-type solution and integrability of eq.(1.61).

1.6 Complex Hamiltonian systems

In the following sections we describe the quantum theory of real, Complex, hermitian and non-hermitian Hamiltonians and various methods of complexifications for Hamiltonians. It is widely accepted that "Hamiltonian operator must be hermitian in order that the energy levels be real and that the theory be unitary". This was the assumption and one is taught at the introductory courses on quantum mechanics. Then it is commonly believed that the Hamiltonian must be Hermitian in order to ensure that the energy spectrum (the eigenvalues of the Hamiltonian) is real and that the time evolution of the theory is unitary (probability is conserved in time). The mathematical expression for the hermiticity of a Hamiltonian is written $H = H^\dagger$, where the symbol '†' denotes the usual Dirac hermitian conjugation; that is, transpose and complex conjugate. In 1998 this false impression has been challenged by Bender and Boettcher [52] who surprised the physics community by showing an experimental evidences and also numerically supported conjecture that quite a few one-dimensional quantum potentials $V(x)$ may generate bound states $\psi(x)$ with real energies E_n even when the potentials themselves are not real. They show that because $\mathcal{PT}$- symmetry is an alternative condition to Hermiticity, many new Hamiltonians can be constructed that would have been rejected in the past because they are not Hermitian. Recently Bender *et al* have discussed class of complex Hamiltonians (usually called as the $\mathcal{PT}$-symmetric Hamiltoinians) [49]. In the past few years it has been recognized that the requirement of Hermiticity, which is often stated as an fundamental law of quantum mechanics, may be replaced by the mathematical and more physical requirement of space-time reflection symmetry ($\mathcal{PT}$- symmetry) without losing any of the essential physical features of quantum mechanics. Theories defined by non-hermitian $\mathcal{PT}$- symmetric Hamiltonians exhibit exceptional and unexpected properties at the classical as well as at the quantum level. In the following section we will bring out how the requirement of hermiticity can be bypassed and discusses the properties of some non-hermitian $\mathcal{PT}$- symmetric quantum theories. The $\mathcal{PT}$- symmetric transformation or adaption of the non-hermitian Hamiltonians exhibits real eigenvalues provided that $\mathcal{PT}$- symmetry is unbroken. Applications to hermitian and non-hermitian Hamiltonians are also discussed in the end of this chapter.

1.7 Hermiticity: the usual definition

The real and positive eigenvalues of many Hamiltonians (newly known as $\mathcal{PT}$-symmetric Hamiltoinians) have been discussed [49] sets-up a question that, does a non-hermitian Hamiltonian define a physical theory of quantum mechanics or is the reality and positivity of the spectrum merely a facinitating mathematical curiosity exhibited by some special classes of complex eigenvalue problems. It must be assured that the physical quantum theory must have bounded energy spectrum, acquire a Hilbert space of state vectors which are confer with an inner product having a positive norm and unitary time evolution. The simplest condition on the Hamiltonian H which guarantees that the quantum theory satisfies these requirements is that the H be real and symmetric. However, this condition has certain restriction. The condition $H = H^\dagger$ is allowed or justified to be the complex Hamiltonian, as long as it is Dirac hermitian. In the last section of this chapter it is described that how one can replace the condition of hermiticity by the condition that H have an unbroken $\mathcal{PT}$- symmetry and still satisfies the above requirements for a physical quantum theory. For this purpose, we summarize the procedure for analyzing a theory defined by a conventional hermitian quantum mechanical Hamiltonian. The procedure is given below:

1. Eigenfunctions and eigenvalues of H:- For a given Hamiltonian H, one can write down the time-independent Schrödinger equation associated with H and calculate the eigenfunctions $\Psi_n(x)$ and eigenvalues E_n. Usually the calculations of eigenvalues and eigenfunctions are performed numerically, since it is difficult to carry out calculations analytically.

2. Orthogonality of eigenfunctions:- The eigenfunctions of H will be orthogonal w.r.t. the standard hermitian inner product

$$(\Psi, \Phi) \equiv \int \Psi(x)^* \Phi(x) dx. \tag{1.63}$$

Orthogonality means that the inner product of two eigenfunctions of H say $\Psi_m(x)$ and $\Phi_n(x)$ associated with two different eigenvalues E_m and E_n vanishes

$$(\Psi_m, \Phi_n) = 0. \tag{1.64}$$

3. Orthonormality of eigenfunctions:- Since the Hamiltonian is hermitian, the norm of any vector is guaranteed to be positive. This means that we can normalize the eigenfunctions of H so that the norm of every eigenfunction is unity then

$$(\Psi_n, \Psi_n) = 1. \tag{1.65}$$

4. Time evolution and unitarity:- For a hermitian Hamiltonian the time evolution operator $e^{-i\mathbf{Ht}}$ is unitary and it automatically preserves the inner product

$$\Big(\chi(\mathbf{t}), \chi(\mathbf{t})\Big) = \Big(\chi(0)\, e^{i\mathbf{H}t}, e^{-i\mathbf{H}t}\chi(0)\Big) = \Big(\chi(0), \chi(0)\Big). \tag{1.66}$$

5. Completeness of eigenfunctions:- It is a theorem in the Hilbert space for linear operators which states that any (finite norm) vector χ can be expressed as a linear combination of eigenfunctions of H

$$\chi = \sum_{n=0}^{\infty} a_n \Psi_n. \tag{1.67}$$

In other words, we can say that the eigenfunctions of a hermitian Hamiltonian are complete. The formal statement of completeness in co-ordinate space is the reconstruction of the unit operator as a sum over the eigenfunctions

$$\sum_{n=0}^{\infty} [\Psi_n(x)]^* \Psi_n(y) = \delta(x - y). \tag{1.68}$$

6. Observable:- An observable is represented by a linear hermitian operator. The outcome of a measurement is one of the real eigenvalues of this operator. The other topics such as classical and semiclassical limit of quantum theory, probability and current density for perturbative and non-perturbative calculations can also be considered for hermitian Hamiltonians.

1.7.1 The Complex Hamiltonian and various way of its complexification

A complex form of the Hamiltonian is expected to offer some clue to some of these additional properties. The use of a complex potential in the context of the optical model of a nucleus [50] has been known for more than 50 years now. In recent years, there has appeared several newly discovered [1] phenomena in physics and chemistry (for example, the phenomena pertaining to resonance scattering in atomic, molecular, and nuclear physics and to some chemical reactions as well) whose theoretical understanding might require the use of complex Hamiltonians. Complex Hamiltonians (or their variants in the form of dynamical invariants) have also been studied in several other theoretical contexts. For example, the studies of complex trajectories with regard to the calculation of a semiclassical coherent-state propagator in the path integral method have attracted particular interest in laser physics [51].

The complex Hamiltonian systems are those in which Hamiltonians are non-hermitian and corresponding eigenvalues are complex.

Several years ago, Bessis (in a private communication with J. Zinn-Justin, who was studying Lee-Yang singularities using renormalization group methods) conjectured on the basis of numerical studies that the spectrum of the Hamiltonian

$$H = p^2 + x^2 + ix^3$$

is real and positive [1]. But there is no rigorous proof this conjecture unto late nineties. Thus a complex Hamiltonian does not guarantee for real eigenvalues. In 1997 C.M. Bender and S. Boettcher [52] (See also M. Znojil [83]) used the non-Hermitian Hamilton operator $H = p^2 + x^2 + ix = p^2 + (x + i/2)^2 + 1/4$ obtained from a Harmonic Oscillator shifted to a complex space point $x = -i/2$ as an example to show that its spectrum $E_n = (2n + 1) + 1/4 = 2n + 5/4$ can be indeed real due to the underlying "$\mathcal{PT}$-symmetry". Complex Hamiltonian might be found in the context of condensed matter physics. Consider the complex crystal lattice whose potential is $V(x) = isinx$. While the Hamiltonian $H = p^2 + isinx$ is not Hermitian, it is $\mathcal{PT}$-symmetric and all of its energy bands are real. In recent years, $\mathcal{PT}$-symmetric Hamiltonians have been discussed [52], in which, despite the lack of conventional hermiticity of H, the eigenvalue spectrum for certain domains of underlying parameter of the system turns out to be real . Hollowood [53] through the Hamiltonian of a complex Toda lattice and J.Zinn-Justin [54] showed that, though the Hamiltonian is non-Hermitian, the energy levels are real. Then it is argued that the reality of the spectrum is a consequence of the combined action of parity and time reversal invariance of H. We discuss two complex Hamiltonian systems as discussed by Bender [52] in his paper

$$H = p^2 + ix^3 + ix; \quad H = p^2 + ix^3 + x \tag{1.69}$$

For example, the Hamiltonian 1st one is $\mathcal{PT}$-symmetry and its entire spectrum is positive definite but the second Hamiltonian is not $\mathcal{PT}$-symmetric, and the entire spectrum is complex.

Mathematically, the Hamiltonian is expressed generally as the sum of kinetic energy and potential energy in one dimension as

$$H(x, p) = \frac{p^2}{2m} + V(x). \tag{1.70}$$

It is to be noted that for a physical system while the form of the first term is reserved, but the second term $V(x)$, can have different functional forms depending upon the system under study. In literature, there exist various way (although some of them are interconnected) of complexifying a given Hamiltonian, which are described as follows:

Method-1:- The Hamiltonian can be made complex by converting a real two dimensional phase space (x-p plane) into a complex x-p plane (z-plane). If one defines

$$z = p + i\omega_0 x, \qquad z^* = p - i\omega_0 x,$$

and writes $H(x,p)$ as $H(z,z^*)$ for real x,p and ω_0. While such a choice has been used in studying the second quantized adaption of the harmonic oscillator in field-theory or the Heisenberg quantum mechanics of the oscillator [50]. It is expected to work well for even-powered potentials. In spite of complexifying the phase plane generated by real x and p, the reality of the Hamiltonian can be retained for this case which is not the case of odd-powered potentials.

Method-2:- In this method of complexification, one defines two independent complex variables as $u = x/b + ip/c$ and $v = x/b - ip/c$ where b and c are treated as complex numbers which give rise to a variety of possibilities regarding the nature of the transformation from (x,p) space to (u,v) space and subsequently that on reality of H. In this case the u, u^*, v, v^* become the new degrees of freedom for describing the system. If $b = b^*$ and $c = c^*$ then u and v become complex conjugate pairs [55].

Method-3:- Another method of complexifying the Hamiltonian in which one replaces $x \to x\exp(i\theta)$ and writes the complex scaled Hamiltonian operator H_θ as

$$H_\theta = S^{-1}(\theta)\hat{H}S(\theta),$$

where S is the complex scaling operator defined by $S = exp\Big[i\theta x(d/dx)\Big]$ such that $Sf(x) = f(xe^{i\theta})$ for an analytical function $f(x)$. The complex scaling associates the resonance phenomena, as it appears in atomic, molecular or nuclear physics [56], with a square integrable function, rather than with a collection of continuum eigenstates of the unscaled hermitian Hamiltonian.

For the quantum system $\left(\text{where } p^2 \to -\hbar^2 \frac{\partial^2}{\partial x^2}\right)$ one obtains

$$H_\theta = -\frac{1}{2}e^{-2\theta}\frac{\partial^2}{\partial x'^2} + V(xe^{i\theta}), \tag{1.71}$$

where $x = x'\exp(i\theta)$ and $\hbar = m = 1$.

Method-4:- The Hamiltonian in eq.(1.70) can be made complex by just considering the parameters in $V(x)$ as complex, including the mass parameter m as well, however x and p are kept as real [57].

Method-5:- The Hamiltonian can be made complex by assuming only the potential function $V(x)$ as complex [81], say $V(x) = V_r(x) + iV_i(x)$. The simplest choice for this case is the complex square-well potential, namely $V(x) = V_0 + iW_0$, used by Feshbach et.al [50] in the optical model of the nucleus.

Method-6:- The Hamiltonian $H(x,p)$ can be made complex by considering each physical variable x and p as complex, i.e, by extending the real two dimensional phase space (x,p) to a complex phase space with four degrees of freedom. This is possible by writing x and p in either of the way, namely

$$x = x_1 + ix_2; \qquad p = p_1 + ip_2 \tag{1.72}$$

where (x_1, p_1) and (x_2, p_2) are real and considered as canonical pairs [56]. In this way, one can transform a real phase space (x, p) into a complex phase space (x_1, p_2, p_1, x_2) with two additional degrees of freedom namely x_2 and p_2. The complex phase space [57] characterized by the equation for x and p are known as the extended complex phase space (ECPS) approach. In this approach both x and p are separately made complex by extending each of them to the corresponding complex planes i.e. inserting an imaginary component in each.

The transformation in eq.(1.72) is used by Rao *et.al* [58] in their study of ion-acoustic waves in plasma and more recently by Yang [59] in developing a complex mechanics of which conventional classical and quantum mechanics may appear as special cases. The transformation in eq.(1.72) is used by Xavier and de Aguiar [60] in their coherent state studies and by Kaushal *et.al* [61, 62] in studying certain aspects of classical and quantum mechanics of non-hermitian systems.

Method-7:- Another class of complex Hamiltonians called the $\mathcal{PT}$- symmetric Hamiltonians [49, 52] in which the Hamiltonian is non-hermitian but eigenvalue spectrum, for certain parametric domain, is real and this reality of the spectrum is a consequence of the combined action of parity and time reversal invariance of H. The parity operator $\hat{\mathcal{P}}$ and time reversal operator $\hat{\mathcal{T}}$ are defined by their action on position and momentum operators as

$$\hat{\mathcal{P}} : \hat{x} \longrightarrow -\hat{x};\ \hat{p} \longrightarrow -\hat{p}, \qquad \hat{\mathcal{T}} : \hat{x} \longrightarrow \hat{x};\ \hat{p} \longrightarrow -\hat{p};\ \hat{i} \longrightarrow -\hat{i}. \tag{1.73}$$

However, the combined parity and time, $(\hat{\mathcal{P}}\hat{\mathcal{T}})$, operator has the following effects

$$\hat{\mathcal{P}}\hat{\mathcal{T}} : \qquad \hat{x} \longrightarrow -\hat{x} \quad ; \quad \hat{p} \longrightarrow \hat{p} \quad ; \quad \hat{i} \longrightarrow -\hat{i}. \tag{1.74}$$

Note that $\hat{\mathcal{T}}$ changes the sign of i because, like the parity operator, it preserves the fundamental commutation relation of quantum mechanics, $[x, p] = i$, known as the Heisenberg algebra. Clearly, from these definitions, we obtain the following identity

$$\hat{\mathcal{P}}^2 = \hat{\mathcal{T}}^2 = 1. \tag{1.75}$$

In $\mathcal{PT}$-symmetric quantum mechanics, it is not necessary for a Hamiltonian to be invariant under either $\mathcal{P}$ or $\mathcal{T}$ individually, it may be invariant under combined operation $(\hat{\mathcal{P}}\hat{\mathcal{T}})$. This must also be true if we were to have a complex position and momentum, in which eqs.(1.73) and (1.74) would respectively become:

$$\begin{aligned} &\hat{\mathcal{P}} : Re\ \hat{x} \longrightarrow -Re\ \hat{x},\ Im\ \hat{x} \longrightarrow -Im\ \hat{x}; \quad Re\ \hat{p} \longrightarrow -Re\ \hat{p},\ Im\ \hat{p} \longrightarrow -Im\ \hat{p}, \\ &\hat{\mathcal{T}} : Re\ \hat{x} \longrightarrow Re\ \hat{x},\ Im\ \hat{x} \longrightarrow -Im\ \hat{x}; \quad Re\ \hat{p} \longrightarrow -Re\ \hat{p},\ Im\ \hat{p} \longrightarrow Im\ \hat{p}. \end{aligned} \tag{1.76}$$

If we define $x_1 \equiv Re\ x, \quad p_2 \equiv Im\ x, \quad p_1 \equiv Re\ p, \quad x_2 \equiv Im\ p$, the complex versions of x and p are written as $x = x_1 + ip_2, \quad p = p_1 + ix_2$; where (x_1, p_1) and (x_2, p_2) are real and considered as canonical pairs [73]. In term of new notations for $\hat{\mathcal{P}}$ and $\hat{\mathcal{T}}$ invariance simply imply that

$$\hat{\mathcal{P}} : (x_1, p_2, p_1, x_2) \rightarrow (-x_1, -p_2, -p_1, -x_2); \quad \hat{\mathcal{T}} : (x_1, p_2, p_1, x_2) \rightarrow (x_1, -p_2, -p_1, x_2), \quad i \rightarrow -i;$$

$$\hat{\mathcal{P}}\hat{\mathcal{T}} : (x_1, p_2, p_1, x_2) \longrightarrow (-x_1, p_2, p_1, -x_2), \quad i \longrightarrow -i.$$

With regard to these methods of complexification the following remarks are in order:

1. Type of the complexification inducted by eq.(1.72) brings in two additional variables (x_2, p_2) in hyperspace which may or may not have any link with the spatio-temporal cartesian grid. They however follow the same rule of the game as the physical variables (x_1, p_1). Thus the variables (x_1, p_1) and their functions will account for the physical reality in the nature, the variables (x_2, p_2) and their functions will account for a reality that is beyond the physical reality in the nature i.e. for the spurious effect.

2. It can be argued that some of the approaches followed recently [58, 62] to study the real eigenvalue spectra of non-hermitian Hamiltonians are based either on the parametric complexification of H or are the particular cases of method-3.

3. The form of the potential function $V(x)$ is responsible for the nature of the trajectory of the particle whereas the finer details of this trajectory are taken care of by the parameters appearing in it. Therefore, any complexification of parameters cannot lead to a basic change in the nature of the trajectory except for allowing some variations around its main track. In some sense the parametric complexification of H affects the geometrical properties but not the geometry of the space-time structure in nature.

It is worth to mention that while method-3 above deals with the complexification of the coordinates alone and that of the momentum follows from it. In methods-1 and 2 the complex phase space is obtained by assuming the variables x and p as real and complexity arises from the parameter space. On the other hand, methods-3 and 7 both x and p are separately made complex by extending each of them to the corresponding complex planes, i.e, by way of inserting imaginary component in each. Further, the number of the degrees of the freedom do not change in methods-1 and 2 except for parameter space, becoming complex, whereas they get doubled in methods-3 and 7. In the latter cases the (x, p) real phase plane is now replaced by a complex phase space (x_1, p_2, p_1, x_2) with two additional degrees of freedom namely x_2 and p_2, the imaginary part of x is identified by p_2 and that of p by x_2 in eq.(1.72). From physics point of view x_2 and p_2 are fictitious/spurious components of momentum and coordinate and their presence in eq.(1.72) as such allows the introduction of some sort of coordinate momentum coupling in the dynamical system. For such an interpretation of imaginary part of x and p, one needs to modify eq.(1.72) in the form

$$x = x_1 + idp_2, \qquad p = p_1 + id^{-1}x_2. \tag{1.77}$$

for the dimensional considerations. In the present study '$d = 1$' is selected.

A consistent physical theory of quantum mechanics can be built on a complex Hamiltonian that is non-hermitian but instead satisfies the physical condition of space-time reflection symmetry ($\mathcal{PT}$- symmetry). The transformation in eq.(1.72) has been used by Xavier and de-Aguiar to develop an algorithm for the computation of the semi-classical coherent-state propagator. This approach is used in earlier chapter-3 and chapter-4 for studying the two dimensional complex systems.

1.8 Concept of $\mathcal{PT}$- symmetry

The central idea of $\mathcal{PT}$-symmetric quantum theory is to replace the condition that the Hamiltonian of a quantum theory be Hermitian with the weaker condition that it possess space-time reflection symmetry ($\mathcal{PT}$-symmetry). This allows one to construct and study many new kinds of Hamiltonians that would previously have been ignored. These new Hamiltonians have remarkable mathematical properties and it may well turn out that these new Hamiltonians will be useful in describing the physical world. It is crucial, of course, that in replacing the condition of Hermiticity by $\mathcal{PT}$-symmetry we do not give up any of the key physical properties that a quantum theory must have. We will see that if the $\mathcal{PT}$- symmetry of the Hamiltonian is not broken, then the Hamiltonian will exhibit all of the features of a quantum theory described by a Hermitian Hamiltonian. (The word broken as used here is a technical term that will be explained in following section) we begin by reviewing some basic ideas of quantum theory. For simplicity, in this chapter we restrict our attention to one-dimensional quantum-mechanical systems. Also, we work in units where Plancks constant $\hbar = 1$. In elementary courses on quantum mechanics one learns that a quantum theory is specified by the Hamiltonian operator that acts on a Hilbert space. The Hamiltonian H does three things:

1. The Hamiltonian determines the energy eigenstates $|E_n\rangle$. These states are the eigenstates of the Hamiltonian operator and they solve the time-independent Schrodinger equation $H|E_n\rangle = En|E_n\rangle$. The energy eigenstates span the Hilbert space of physical state vectors. The eigenvalues E_n are the energy levels of the quantum theory. In principle, one can observe or measure these energy levels. The outcome of such a physical measurement is a real number, so it is essential that these energy eigenvalues be real.

2. The Hamiltonian H determines the time evolution in the theory. States $|t\rangle$ in the Schrödinger picture evolve in time according to the time-dependent Schrödinger equation $H|t\rangle = -i\frac{d}{dt}|t\rangle$, whose formal solution is $|t\rangle = e^{iHt}|0\rangle$. Operators $A(t)$ in the Heisenberg picture evolve according to

the time-dependent Schrödinger equation $\frac{d}{dt}A(t) = -i[A(t), H]$, whose formal solution is $A(t) = e^{iHt}A(0)e^{-iHt}$.

3. The Hamiltonian incorporates the symmetries of the theory. A quantum theory may have two kinds of symmetries: continuous symmetries, such as Lorentz invariance, and discrete symmetries, such as parity invariance and time reversal invariance. A quantum theory is symmetric under a transformation represented by an operator A if A commutes with the Hamiltonian that describes the quantum theory: $[A, H] = 0$. Note that if a symmetry transformation is represented by a linear operator A and if A commutes with the Hamiltonian, then the eigenstates of H are also eigenstates of A. Two important discrete symmetry operators are parity (space reflection), which is represented by the symbol $\mathcal{P}$, and time reversal, which is represented by the symbol $\mathcal{P}$. The operators $\mathcal{P}$ and $\mathcal{T}$ are defined by their effects on the dynamical variables $\hat{x}$ (the position operator) and $\hat{p}$ (the momentum operator). The operator $\mathcal{P}$ is linear and has the effect of changing the sign of the momentum operator $\hat{p}$ and the position operator $\hat{x} : \hat{p} \to -\hat{p} \quad and \quad \hat{x} \to -\hat{x}$. The operator $\mathcal{T}$ is antilinear and has the effect $\hat{p} \to -\hat{p}, \quad \hat{x} \to \hat{x}, \quad and \quad i \to -i$. Note that $\mathcal{P}$ changes the sign of i because (like $\mathcal{P}$) $\mathcal{T}$ is required to preserve the fundamental commutation relation $[\hat{x}, \hat{p}] = i$ of the dynamical variables in quantum mechanics.

Quantum mechanics is an association between states in a mathematical Hilbert space and experimentally measurable probabilities. The norm of a vector in the Hilbert space must be positive because this norm is a probability and a probability must be real and positive. Furthermore, the inner product between any two different vectors in the Hilbert space must be constant in time because probability is conserved. The requirement that the probability not change with time is called unitarity. Unitarity is a fundamental property of any quantum theory and must not be violated. To summarize the discussion so far, the two crucial properties of any quantum theory are that the energy levels must be real and that the time evolution must be unitary. There is a simple mathematical condition on the Hamiltonian that guarantees the reality of the energy eigenvalues and the unitarity of the time evolution; namely, that the Hamiltonian be real and symmetric. To explain the term symmetric, as it is used here, let us first consider the possibility that the quantum system has only a finite number of states. In this case the

Hamiltonian is a finite-dimensional symmetric matrix

$$H = \begin{pmatrix} a & b & c & . & . & . \\ b & d & e & . & . & . \\ c & e & f & . & . & . \\ . & . & . & . & & \\ . & . & . & & . & \\ . & . & . & & & . \end{pmatrix}$$

whose entries $a, b, c, d, e, f, ,$ are real numbers. For systems having an infinite number of states we express H in coordinate space in terms of the dynamical variables $\hat{x}$ and $\hat{p}$. The $\hat{x}$ operator in coordinate space is a real and symmetric diagonal matrix, all of whose entries are the real number x. The $\hat{p}$ operator in coordinate space is imaginary and anti-symmetric because $\hat{p} = -i\frac{d}{dx}$ when it acts to the right but, as we can see using integration by parts, $\hat{p}$ changes sign $\hat{p} = i\frac{d}{dx}$ when it acts to the left. The operator $\hat{p^2} = -\frac{d^2}{dx^2}$ is real and symmetric. Thus, any Hamiltonian of the form $H = \hat{p^2} + V(\hat{x})$ when written in coordinate space is real and symmetric. However, the condition that H be real and symmetric is not the most general condition that guarantees the reality of the energy levels and the unitarity of the time evolution because it excludes the possibility that the Hamiltonian matrix might be complex. Indeed, there are many physical applications which require that the Hamiltonian be complex. There is a more general condition that guarantees spectral reality and unitary time evolution and which includes real, symmetric Hamiltonians as a special case. This condition is known as Hermiticity. The condition that H must exhibit Dirac Hermiticity is often taught as an axiom of quantum mechanics. The Hamiltonians $H = \hat{p^2} + \hat{p} + V(\hat{x})$ and $H = \hat{p^2} + \hat{px} + \hat{xp} + V(\hat{x})$ are complex and nonsymmetric but they are Hermitian. Bender *et al* show that while Hermiticity is sufficient to guarantee the two essential properties of quantum mechanics, it is not necessary. They describe an alternative way to construct complex Hamiltonians that still guarantees the reality of the eigenvalues and the unitarity of time evolution and which also includes real, symmetric Hamiltonians as a special case. they maintain the symmetry of the Hamiltonians in coordinate space, but allow the matrix elements to become complex in such a way that the condition of space-time reflection symmetry is preserved. The new kinds of Hamiltonians are discussed in this chapter are symmetric and have the property that they commute with the $\mathcal{PT}$ operator: $[H, PT] = 0$. In analogy with the property of Hermiticity $H = H^{\dagger}$, Bender *et al* express the property that a Hamiltonian is $\mathcal{PT}$- symmetric by using the notation $H = H^{\mathcal{PT}}$. New kinds of complex Hamiltonians are symmetric in coordinate space but are not Hermitian in the Dirac sense. To reiterate, acceptable complex Hamiltonians may be either Hermitian $H = H^{\dagger}$ or $\mathcal{PT}$- symmetric $H = H^{PT}$, but not both. Real symmetric Hamiltonians may be both Hermitian and $\mathcal{PT}$- symmetric. Using $\mathcal{PT}$- symmetry as an alternative condition to Hermiticity, one can construct infinitely many new Hamiltonians that would have been rejected in the past because they are not Hermitian. An example of such a $\mathcal{PT}$- symmetric Hamiltonian is

$$H = \hat{p^2} + i\hat{x^3}$$

It is easy to construct infinitely many Hamiltonians that are not Hermitian but do possess PT symmetry. For example, consider the one-parameter family of Hamiltonians

$$H = p^2 + x^2(ix)^{\epsilon}.$$

where ϵ is real

Note that while H in (above eqn.) is not symmetric under $\mathcal{P}$ or $\mathcal{T}$ separately, it is invariant under their combined operation. We say that such Hamiltonians possess space-time reflection symmetry. Other examples of complex Hamiltonians having $\mathcal{PT}$- symmetry are $H = p^2 + x^4(ix)^\epsilon$, $H = p^2 + x^6(ix)^\varepsilon$, and so on [49, 58]. The class of $\mathcal{PT}$- symmetric Hamiltonians is larger than and includes real symmetric Hermitian because any real symmetric Hamiltonian is automatically $\mathcal{PT}$-symmetric. For example, consider the real symmetric Hamiltonian $H = p^2 + x^2 + 2x$. This Hamiltonian is time-reversal symmetric, but according to the usual definition of space reflection for which $x \to -x$, this Hamiltonian appears not to have $\mathcal{PT}$-symmetry. However, recall that the parity operator is defined only up to unitary equivalence [4]. In this example, if we express the Hamiltonian in the form $H = p^2 + (x+1)^2 - 1$, then it is evident that H is $\mathcal{PT}$- symmetric, provided that the parity operator performs a space reflection about the point $x = -1$ rather than $x = 0$. See Ref. [59] for the general construction of the relevant parity operator.

1.8.1 History of non-hermitian Hamiltonian

T. T. Wu in 1959, published a paper calculating the ground state energy of "Bose sphere" [63]. In this paper, the common problem was the divergence of the ground state energy. Wu found that by using a non-diagonalize and non-hermitian Hamiltonian, this problem can be avoided. Remarkably, Hamiltonian possessed real eigenvalues. However, the paper offered no justification for introducing such a Hamiltonian, other than it gave the required solution. In particular, it gave real numbers representing low lying energy levels of a Bose system. This paper was the earliest in which use of non-hermitian Hamiltonians found in the literature survey.

J.Wong [64] 1967 published a paper about physically reasonable non-hermitian Hamiltonians [**?**]. He made the point that Hamiltonians of closed systems are hermitian, but that when an external interaction is considered, the Hamiltonian losses its hermiticity. Therefore, a class of physically reasonable Hamiltonians would be perturbed type

$$\hat{H} = \hat{H}_0 + \hat{H}^{(1)},$$

where $\hat{H}^{(1)} = g\hat{H}$ is the perturbation term, $\hat{H}_0$ is hermitian Hamiltonian and $\hat{H}$ is non-hermitian Hamiltonian and 'g' is simply a parameter to vary the influence of $\hat{H}$. Also $\hat{H}$ has the restriction that it may have only discrete spectrum unless part of its spectrum coincides with that of $\hat{H}_0$. He calls this class as dissipative but does not fully define the term. The paper derived several propositions relating to Hamiltonian in this class. However, at no point in the paper [63], the reality of the spectrum is mentioned. Moreover, complex eigenvalues seem to be admitted, yet there is no explanation of how they could possibly be physically reasonable. T. T. Wu and Bender (in 1969) discusses an example of the anharmonic oscillator [65]. This paper does not explicitly discuss hermitian or non-hermitian Hamiltonians. The Hamiltonian for such a model is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}\mu^2\hat{x}^2 - g\hat{x}^4. \tag{1.78}$$

In above equation the parameter g is extended into the complex plane, so the non-hermiticity of the Hamiltonian is implicit. Coincidentally, this approach was taken because perturbation method of calculating the ground state of this system were also divergent but later on renormalized perturbation theory cured this problem. Haydock and Kelly (in 1975) published a letter reporting that the recursive method with non-hermitian Hamiltonians to calculate the electronic structure of crystalline Arsenic [66]. According to this letter, chemical pseudo-potential theory often give rise to a non-hermitian representation of interactions between localized electron orbits [69]. This was one of the earliest paper to say explicitly that the condition of hermiticity was sufficient but not necessary to ensure a real eigen spectrum. G. E. Stedman and P. H. Butler (n 1980) published a paper reviewing material in the field of time reversal and point group theory, in particular, focussing on the effect and selection rules for time reversal in the rotation group $so(3)$ [67]. This was the earliest reference found to the term *"time reversal operator"* in relation to complex conjugation, as it is used in $\mathcal{PT}/\mathcal{CPT}$ symmetry. F. Faisal and J. Moloney (in 1981) studied a quantum decay process with a non-hermitian Hamiltonian and Schrödinger wave equation [68]. They claimed that, as a consequence of the uncertainty principle, a decaying state could not have a sharp energy, and that the width of such an energy level could be represented by an imaginary energy component. Furthermore, such complex energies can be shown to be eigenvalues of non-hermitian Hamiltonian associated with the decay process. This paper also includes a non-unitary yet self-consistent (probability conserving) algorithm for time evolution. These are just some examples of work from last four decades, which uses non-hermitian theories. The common factor in all these papers is that such theories were introduced on phenomenological or heuristic grounds. In other words, such theories were chosen because they fitted with experimental observations. From a pragmatic point of view, merely fitting theories to observations could be considered reasonable. After all, nature is always correct! However, papers which allowed the use of a complex spectrum never appear to justify or elucidate the validity of such spectra. Therefore, since the prediction of eigenvalues is crucial link between theory and experimental observations [50], it would seem highly questionable to allow eigenvalues which could not be physically measured. However, Hatano and Nelson justified the use of a complex spectrum [51]. There are the following requirements for a physically acceptable non-hermitian theory:

- The eigen spectrum must be entirely real.
- Time evolution must be unitary, to prevent *"probability leakage"*[78].
- The set of eigenstates must be complete.

1.8.2 Non-hermitian Hamiltonians: the theory

In this section we follow the same prescription as in previous section to study a non-hermitian Hamiltonian having unbroken $\mathcal{PT}$- symmetry. For definiteness, we do not know the definition of inner product, as we do in the case of ordinary hermitian quantum mechanics. We will have to discover the correct inner product in the course of our analysis. The inner product is determined by the Hamiltonian itself. The procedure as follows:

1. Eigenvalues and eigenfunctions of H:- Here we assume that the eigenvalues (E_n) can be found by using analytical or numerical methods. These eigenvalues are all real, which is equivalent to assuming the $\mathcal{PT}$- symmetry of H is unbroken; i.e., all the eigenfunctions of H are also the eigenfunctions of $\mathcal{PT}$ operator. The zeros of $\mathcal{PT}$- symmetric eigenfunctions have interesting complex interlacing properties [70].

2. Orthogonality of eigenfunctions:- To test the orthogonality of the eigenfunctions, we must specify an inner product. Since we do not yet know what inner product to use, we might try to guess an inner product. By analogy, one might think that since the inner product is appropriate for hermitian Hamiltonian ($H = H^\dagger$), a good choice for an inner product associated with a $\mathcal{PT}$- symmetric Hamiltonian ($H = H^{\mathcal{PT}}$) might be

$$(\Psi, \Phi) \equiv \int_c dx[\Psi(x)]^{\mathcal{PT}}\Phi(x) = \int_c dx[\Psi(-x)]^*\Phi(x), \tag{1.79}$$

where c is a contour. With this inner product definition one can show by a trivial integration by parts using time-independent SE that pairs of eigenfunctions of H associated with different eigenvalues are orthogonal. However, this guess for an inner product is not acceptable for formulating a valid quantum theory because the norm of a state is not necessarily positive.

3. $\mathcal{PT}$- symmetric normalization:- We know that the eigenfunctions $\Psi_n(x)$ of H are also eigenfunctions of the $\mathcal{PT}$ operator with eigenvalue $\lambda = e^{i\alpha}$, where λ and α depend on 'n'. Thus, we can construct $\mathcal{PT}$-normalized eigenfunctions $\Phi_n(x)$ defined by

$$\Phi_n(x) \equiv e^{-i\alpha/2}\Psi(x). \tag{1.80}$$

By this construction, $\Phi_n(x)$ is still an eigenfunction of H and it is also an eigenfunction of $\mathcal{PT}$ operator with eigenvalue '1'. One can also show both numerically and analytically that the algebraic sign of the $\mathcal{PT}$ norm in (1.87) of $\Phi_n(x)$ is $(-1)^n$ for all n and for all values of $\epsilon > 0$. Thus, we define the eigenfunctions so that their $\mathcal{PT}$ norms are exactly $(-1)^n$

$$\int_c dx[\Phi_n(x)]^{\mathcal{PT}}\Phi_n(x) = \int_c dx[\Phi_n(-x)]^*\Phi_n(x) = (-1)^n, \tag{1.81}$$

where c is a contour. In terms of these $\mathcal{PT}$ normalized eigenfunctions there is a simple but unusual statement of completeness

$$\sum_{n=0}^{\infty}(-1)^n\Phi_n(x)\Phi_n(y) = \delta(x-y). \tag{1.82}$$

This statement of completeness has been verified both numerically and analytically for all $\epsilon > 0$.

4. The $\mathcal{CPT}$ inner Product:- To construct an inner product with a positive norm for a complex non-hermitian Hamiltonian having an unbroken $\mathcal{PT}$- symmetry, we will construct a new linear operator $\mathcal{C}$ that commutes with both H and $\mathcal{PT}$. Because $\mathcal{C}$ commutes with the Hamiltonian, it represents a symmetry of H. We denote this symmetry by the symbol $\mathcal{C}$ because the properties of $\mathcal{C}$ are similar to those of the charge conjugation operator in particle physics [55]. The inner product w.r.t. $\mathcal{CPT}$ conjugation is as

$$\langle\Psi|\chi\rangle^{CPT} = \int dx\ \Psi^{CPT}(x)\ \chi(x), \tag{1.83}$$

where $\Psi^{\mathcal{CPT}}(x) = \int dy\ \mathcal{C}(x,y)\ \Psi^*(-y)$. One can show that this inner product satisfies the requirements for the quantum theory defined by H to have a Hilbert space with a positive norm and to be a unitary. The $\mathcal{C}$ operator is represented as a sum over the eigenfunctions of H, but before doing so one must first show how to normalize these eigenfunctions.

5. Positive norm and unitarity in $\mathcal{PT}$- symmetric quantum mechanics:- Now, we can use the new $\mathcal{CPT}$ inner product defined in eq.(1.83). Since the operator $\mathcal{C}$ has been constructed. Like the $\mathcal{PT}$ inner product, this new inner product is phase independent. Also, because the time evolution operator (as in the ordinary quantum mechanics) is $e^{-i\mathbf{H}t}$ and because $\mathbf{H}$ commutes with $\mathcal{PT}$ and $\mathcal{CPT}$ operators respectively. Both the $\mathcal{PT}$ inner product and the $\mathcal{CPT}$ inner product remain time independent as the states evolve. However, unlike the $\mathcal{PT}$ inner product, the $\mathcal{CPT}$ inner product is positive definite because $\mathcal{C}$ contributes a factor of '-1' when it acts on states with negative $\mathcal{PT}$ norm. In terms of the $\mathcal{CPT}$ conjugate, the completeness condition will be

$$\sum_{n=0}^{\infty} \Phi_n(x)\Big[\mathcal{CPT}, \Phi_n(y)\Big] = \delta(x-y). \tag{1.84}$$

6. Construction of the $\mathcal{C}$-operator:- For any Hamiltonian H having an unbroken $\mathcal{PT}$- symmetry there exist an additional symmetry of H connected with the fact that there are equal number of positive and negative norm states. The linear operator $\mathcal{C}$ that includes this symmetry can be represented in coordinate space as a sum over the $\mathcal{PT}$ normalized eigenfunctions of $\mathcal{PT}$- symmetric Hamiltonians as

$$\mathcal{C}(x,y) = \sum_{n=0}^{\infty} \Phi_n(x)\Phi_n(y). \tag{1.85}$$

We note that this equation is identical to the statement of completeness in eq.(1.82) except that the factor $(-1)^n$ is absent. We can use eqs.(1.81) and (1.82) to verify that the square of $\mathcal{C}$ is unity ($\mathcal{C}^2 = 1$)

$$\int \mathcal{C}(x,y)\mathcal{C}(y,z)dy = \delta(x-z). \tag{1.86}$$

Thus, the eigenvalues of $\mathcal{C}$ are ± 1. Also $\mathcal{C}$ commutes with H. Therefore, since $\mathcal{C}$ is linear, the eigen states of H have definite value of $\mathcal{C}$. Specifically

$$\begin{aligned}\mathcal{C}\ \Phi_n(x) \quad &= \int dy\ \mathcal{C}(x,y)\ \Phi_n(x) = \sum_{m=0}^{\infty} \Phi_m(x) \int dy\ \Phi_m(y)\Phi_n(y) \\ &= (-1)^n \Phi_n(x).\end{aligned} \tag{1.87}$$

This new operator $\mathcal{C}$ resembles the charge-conjugation operator in the quantum-field theory. However, the precise meaning of $\mathcal{C}$ is that it represents the measurement of the sign of the $\mathcal{PT}$ norm in (1.81) of an eigen state. The operator $\mathcal{P}$ and $\mathcal{C}$ are distinct square roots of unity operator $\delta(x-y)$. That is $\mathcal{P}^2 = \mathcal{C}^2 = 1$, but $\mathcal{P} \neq \mathcal{C}$ because $\mathcal{P}$ is real and $\mathcal{C}$ is complex. The parity operator in coordinate space is explicitly real [$\mathcal{P}(x,y)$=$\delta(x+y)$], while the operator $\mathcal{C}(x,y)$ is complex because it is sum of product of complex functions. The two operators $\mathcal{P}$ and $\mathcal{C}$ do not commute. However $\mathcal{C}$ does commutes with $\mathcal{PT}$.

1.8.3 Broken and unbroken $\mathcal{PT}$-symmetry

An alternative formalism of quantum mechanics in which the mathematical axiom of the hermiticity ($H = H^\dagger$) is replaced by the physical transparent condition of space-time reflection ($\mathcal{PT}$)- symmetry i.e. $H = H^{\mathcal{PT}}$ and $H^{\mathcal{PT}} \equiv (\mathcal{PT})H(\mathcal{PT})$. Examples of $\mathcal{PT}$- symmetric non-hermitian Hamiltonians are

$$\hat{H} = \hat{p}^2 + i\hat{x}^3, \qquad \hat{H} \;=\; \hat{p}^2 - \hat{x}^4. \tag{1.88}$$

Amazingly the energy levels of these Hamiltonians are real and positive. These two Hamiltonians specify a unitary time evolution even though they are non-hermitian. The Hamiltonians described in eq.(1.88) are the special cases of the general parametric family of $\mathcal{PT}$- symmetric Hamiltonians

$$\hat{H} = \hat{p}^2 + x^2(ix)^\epsilon, \tag{1.89}$$

where the parameter ϵ is real. These Hamiltonians are all $\mathcal{PT}$- symmetric because they satisfy the condition $H = H^{\mathcal{PT}}$. It is shown by Bender *et.al* in 1998 [49, 52] that when $\epsilon \geq 0$ all the eigenvalues of these Hamiltonians are entirely real and positive, but when $\epsilon < 0$ there are complex eigenvalues. From this Bender concluded that $\epsilon \geq 0$ is the parametric region of the *unbroken* $\mathcal{PT}$- symmetry and that $\epsilon < 0$ is the parametric region of the *broken* $\mathcal{PT}$- symmetry. Thus it is clearly of great importance to understand as to when the $\mathcal{PT}$-symmetry is spontaneously broken and when it remains unbroken. As a first step in that direction, it may be worthwhile to look for some analytically solvable $\mathcal{PT}$-symmetry-invariant potentials and try to understand the spontaneous breaking and nonbreaking of the $\mathcal{PT}$-symmetry- symmetry. The purpose of this note is to study one such example. In particular we consider the system described by the non-Hermitian but $\mathcal{PT}$-invariant Hamiltonian ($\hbar = 2m = 1$).

$$H = p^2 - (\zeta cosh2x - iM)^2,$$

where the parameter ζ is real and parameter M has only integer values. Khare *et al* [69] has shown that the quasi-exactly solvable (QES) eigen values of this Hamiltonian are complex conjugate pairs in case the parameter M is an even integer and that in this case the $\mathcal{PT}$- symmetry is indeed spontaneously broken. On the other hand, when M is an odd integer then the QES eigenvalues of this Hamiltonian are real and precisely in this case the $\mathcal{PT}$-symmetry remains unbroken. There are some various examples discussed application sections (QES Potentials).

1.8.4 Hermitian Hamiltonian vs $\mathcal{PT}$- symmetric Hamiltonian

It is not at all obvious whether a Hamiltonian such as H in (1.89) gives rise to a consistent quantum theory. Indeed, while past investigations of this Hamiltonian have shown that the spectrum is entirely real and positive when $\epsilon \geq= 0$, it appeared that one inevitably encountered the severe problem of dealing with Hilbert spaces. In 2003, Bender *et al* [70] explored how $\mathcal{PT}$- symmetry relates to hermiticity, with an approach based on the following established properties:

- $\hat{P}$ is linear and hermitian.
- $\hat{P}$ and $\hat{H}$ commute: $[\hat{H}, \hat{P}] = 0$ (if $\hat{H}$ is $\hat{P}$ invariant).
- $\hat{P}^2 = 1$.
- The n^{th} eigenstate of $\hat{H}$ is also the n^{th} eigenstate of $\hat{p}$ with eigenvalue, $(-1)^n$.

One can show that $\mathcal{PT}$- symmetry is a generalization of hermiticity by showing that all hermitian Hamiltonian have $\mathcal{PT}$- symmetry.

1.9 Applications of hermitian and non hermitian Hamiltonian

There are some applications of the non-hermitian quantum theories, which we will discuss in following sections:

Applications of Hermitian Quantum Theories

Non-hermitian Hamiltonians have various applications in quantum theories such as:

1. Fibre optics
2. Quantum Chemistry
3. Structural Phase Transition
4. Polaron formation in solid
5. False vacuua in field theory
6. Multiphoton Ionization
7. Theory of Strong Interactions without Gluons

In the previous chapters we have discussed the complex Hamiltonian systems along with some interesting examples to find the Invariants in one- and two-and three-dimensions. The solutions obtained are useful in many way. Some of the specific applications of the non- hermitian quantum theories discussed in this chapters are highlighted. Although the results are obtained in this work for some particular systems, yet their study help to understand many theoretical phenomena in physics and chemistry (structural phase transition, polaron formation in solid, concept of false vacua in field theories, model for various molecules, fibre optics etc.). In what follows, we briefly describe the application of non hermitian theory and summarize the finding of the present work along with some concluding remarks. Finally, we describe the application of this above study.

Applications of non-Hermitian quantum theory

It is not yet known whether non-hermitian $\mathcal{PT}$- symmetric Hamiltonians describe phenomena that can be observed experimentally. However, non-Hermitian $\mathcal{PT}$- symmetric Hamiltonians have already appeared in the literature very often and their remarkable properties have been noticed and used by many authors [71, 72]. Cubic non-hermitian Hamiltonian of the form $H = \hat{p}^2 + i\hat{x}^3$ arises in studies of the Lee-Yang edge singularity [73, 74] and in various $Reggeon\ field - theory$ models [75]. In all these cases a non-hermitian Hamiltonian having a real spectrum appeared mysterious at that time, but now the explanation is simple. In every case the non-hermitian Hamiltonian is $\mathcal{PT}$-symmetric and is constructed so that the position operator $\hat{x}$ or the field operator $\hat{\Phi}$ is always multiplied by 'i'. Hamiltonians having $\mathcal{PT}$- symmetry have also been used to describe magnetohydrodynamic systems [76, 77] and to study non-dissipative time-dependent systems interacting with electromagnetic fields [78]. In this section we briefly describe different areas of quantum mechanics in which non-hermitian $\mathcal{PT}$- symmetric Hamiltonians play a crucial and significant role.

1. Quantum Brachistochrone :-The similarity transformation maps the non hermitian $\mathcal{PT}$- symmetric Hamiltonian H to a hermitian Hamiltonian h. The two Hamiltonians, H and h, have the same eigenvalues, but this does not mean they describe the same physics. To illustrate the difference between, H and h, we show how to solve the quantum brachistochrone problem for $\mathcal{PT}$- symmetric and for hermitian quantum mechanics, and we show that the solution to this problem in these two formulations of quantum mechanics is not the same.

2. Complex crystals:- An experimental signal of a complex Hamiltonian might be found in the context of condensed matter physics. Consider the complex crystal lattice whose potential is

$$V(x) = iSinx. \tag{1.90}$$

The optical properties of complex crystal lattices were first studied by Berry and O'Dell, who referred to them as complex diffraction gratings [70]. While the Hamiltonian $H = \hat{p}^2 + isin\ \hat{x}$ is not Hermitian, it is $\mathcal{PT}$- symmetric and all of its energy bands are real. At the edge of the bands the wave function of a particle in such a lattice is bosonic (2π-periodic), and unlike the case of ordinary crystal lattices, the wave function is never fermionic (4π-periodic). The discriminant for a hermitian $sin(x)$ potential is plotted in figure-3 and discriminant for a non-hermitian $i\ sin(x)$ potential is plotted in figure-4. The difference between these two figures is subtle. In figure-4 the discriminant does not go below-2 and thus there are half as many gaps [71]. Direct observation of such a band structure would give unambiguous evidence of a $\mathcal{PT}$- symmetric Hamiltonian. complex periodic potentials having more elaborate band structures have also been found.

3. $\mathcal{PT}$- symmetric quasi-exactly solvable Hamiltonians:- A quantum-mechanical Hamiltonian is said to be quasi-exactly solvable (QES) if a finite portion of its energy spectrum and associated eigenfunctions can be found exactly and in closed form [70]. QES potentials depend on an integer parameter J; for positive values of J one can find exactly the first J eigenvalues and eigenfunctions, typically of a given parity. QES systems are classified using an algebraic approach in which the Hamiltonian is expressed in terms of the generators of a Lie algebra [78]. This approach generalizes the dynamical-symmetry analysis of exactly solvable quantum-mechanical systems whose entire spectrum may be found in closed form by algebraic means.

5. Other applications:- There are some other applications of non-hermitian theories as

1. In the theory of reaction-diffusion systems, many models have been constructed for systems described by matrices that can be non-hermitian [79].

2. Quantum systems characterized by non-hermitian Hamiltonian are of interest in several area of theoretical physics: for example, in Nuclear physics one studies standard Schrodinger Hamiltonian with complex potentials [80], which in this connection are called optical or average nuclear potentials.

3. In the Hatano-Nelson model of non-hermitian magnetic field there is a critical value of the anisotropy (non-hermiticity) parameter below which all eigenvalues are real.

4. Non hermitian interactions are also discussed in field theories, for example, when studying Lee Yang zeros. Even in recent studies on localization-delocalization transition in superconductors [51] and in the theoretical description of diffraction of atoms by standing light waves non-hermitian hamiltonian are of interest

5. In Bose sphere [65] the non-hermitian Hamiltonian represents the low lying energy levels of a system.

6. To study Delocalization transition in condensed matter systems such as vortex flux line deppining in type-II superconductors [52], population biology [51], quantum cosmology, quantum field theory and super symmetric quantum mechanics.

7. Perturbation series for a non-hermitian "effective" Hamiltonian [80] is used in nuclear theory. It also references non hermitian Hamiltonians [52] being used in iterative methods to calculate a wave operator with second order convergence. Optical potential which produces a non Hermitian Hamiltonian with an associated bi-orthogonal basis set. It also references an iterative integration method which can use non Hermitian Hamiltonians. Although the Bloch effective Hamiltonian is non symmetric, yet it can give real eigen values.

8. The non-hermitian operator is used to treat scattering phenomena in atomic and molecular physics [81].

9. The quantum decay process is explained by non hermitian Hamiltonian [82]. As a consequences of uncertainity principle, a decay state could not have sharp energy and that the width of such energy levels could be represented by an imaginary component.

10. It is unknown whether any non-hermitian Hamiltonian can be used to describe any experimentally observable phenomena, although these had already been used to describe interacting systems [65].

11. Non-hermitian Hamiltonian is used to calculate the electronic structure of crystalline Arsenic.The chemical pseudo potential theory often give rise to a non-hermitian representation of interaction between localized electronic orbits .

12. In condensed matter physics Hamiltonians rendered non-Hermitian by an imaginary external field have been introduced to study delocalization transitions in condensed matter systems such as vortex flux-line depinning in type-II superconductors [51, 52].

13. In Bender's first publication on PTSQM he mentioned that non Hermitian theories had even been used for a theoretical model of population Biology.

14. Weigert noted some applications of non-hermitian Hamiltonians in the description of absorptive optical media, inelastic scattering from nuclei and other loss mechanism on the atomic or molecular level. He also noted that non-hermitian theories had recently been "rediscovered" within particle physics.

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Chapter 2

Rationalization Method:Invariants for Classical Systems in ECPS

2.1 Introduction

In physics and mathematics, completely integrable systems, especially in the infinite dimensional setting, are often referred to as exactly solvable models. This obscures the distinction between integrability in the Hamiltonian sense, and the more general classical dynamical systems sense. An imprecise notion of exact solvability as meaning: The existence of sufficient number of invariants, in terms of which the solutions may be expressed. In the general theory of differential systems, there is Frobenius integrability, which refers to over determined systems. In the classical theory of Hamiltonian dynamical systems, there is the notion of Liouville integrability. A classical Hamiltonian $H(x,p)$ is a function from a $2n$-dimensional phase space into the real numbers. Liouville, who defined that a classical Hamiltonian $H(x,p)$ is integrable when it possesses n functionally independent first integrals in involution with a certain degree of regularity, e.g., being smooth or analytic. The complexity of the dynamics defined by $H(x,p)$, i.e., the regular or chaotic behavior of the orbits of the Hamiltonian, strongly depends upon its integrability. Particularly, the dynamics is not considered chaotic when $H(x,p)$ is integrable. We look, therefore, for an answer at the fundamental level of dynamics based on the classification of dynamical systems in terms of instability and integrability.

In the context of differentiable dynamical systems, the notion of integrability refers to the existence of invariant, regular foliations. There is thus a variable notion of the degree of integrability, depending on the dimension of the leaves of the invariant foliation. This concept has a refinement in the case of Hamiltonian systems, known as complete integrability in the sense of Liouville (see section 1.2, chapter 1), which is what is most frequently referred to in this context.

If it becomes possible to obtain all the invariants or constants of motion of a dynamical system, then the system is identified as integrable, otherwise it is a non-integrable one. Also, as mentioned in chapter 1, the number of functionally independent invariants including the Hamiltonian (for autonomous cases) to be same as the dimensionality of the system. For explicitly TD systems (nonautonomous systems) the Hamiltonian is itself not a constant of motion and then one have to find an additional invariant to ensure the integrability of the system. Therefore, one dimensional TID systems are trivially integrable and for two dimensional TID systems one requires to find only one constant of motion to establish it as an integrable system.

After publication of the seminal paper by Bender and Boettcher in 1998 [1], quantum mechanics of non-hermitian complex potentials, particularly $\mathcal{PT}$-symmetric Hamiltonians, is widely studied. But little efforts have been made on the classical front. So construction of classical invariants for non-hermitian systems are also important. Recently, The extended complex phase space (ECPS), characterized by $x = x_1 + ip_2$ and $p = p_1 + ix_2$ are to study some classical aspects of non-hermitian Hamiltoinians. Similar transformations have also been used in some other studies [2, 3, 4, 5]. The ECPS approach was further used in [29] for study of one-dimensional non-hermitian quantum systems. As far as the construction of invariants for non-hermitian systems in an ECPS is concerned, only a few studies of one-dimensional systems have been reported in recent past. Therefore in the present chapter, we obtain invariants for a general classical dynamical system in two-dimensional ECPS.

Although many methods for the construction of real/complex invariants may be found in the literature, here we follow the rationalization method which has been widely used in the past [3, 4, 5]. This method is quite straightforward and can provide exact invariants.

Rationalization method

This method has been used to construct invariants of both TID and TD systems in one and higher dimensions. Whittaker [6] introduced this method in early nineties for the construction of invariants, second order in momenta, of TID systems. Subsequently, this method has been used by several workers for finding invariants of both TID and TD system in one and two dimensions [2, 3, 4, 5, 9, 18]. Since this method is being used for obtaining invariants of two or higher dimensional TID systems, therefore, we briefly describe the method considering a two dimensional TID dynamical system described by the Hamiltonian as

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y), \tag{2.1}$$

Suppose this system admits a constant of motion (a fourth-degree polynomial in momenta) in eq.(2.1), as

$$I = a_0 + a_i\xi_i + \frac{1}{2!}a_{ij}\xi_i\xi_j + \frac{1}{3!}a_{ijk}\xi_i\xi_j\xi_k + \frac{1}{4!}a_{ijkl}\xi_i\xi_j\xi_k\xi_l + ... \tag{2.2}$$

where $i, j, k, l, ... = 1, 2$, $\xi_1 = \dot{x}_1$, $\xi_2 = \dot{x}_2$ and a_0, a_i, a_{ij}, a_{ijk}, a_{ijkl}, etc. are functions of x_1, x_2 only. In fact for TID systems, due to the time-reflection symmetry of Lagrangian, a polynomial constant of motion consists of either even or odd powers in momenta (2.2). Therefore, for the system (2.1) the above expression reduces to.

$$I = a_0 + \frac{1}{2!} a_{ij}\xi_i\xi_j + \frac{1}{4!} a_{ijkl}\xi_i\xi_j\xi_k\xi_l + ... \tag{2.3}$$

The invariance of the function I implies $dI/dt = 0$. i.e.

$$\frac{dI}{dt} = [I, H]_{PB} = 0, \tag{2.4}$$

where $[..]_{PB}$ is Poisson bracket and H is Hamiltonian of system. On rationalizing the expression obtained after using (2.2) in (2.4), with respect to the power of ξ_i, ξ_j, ξ_k... etc. and their all possible products, we get a system of over-determined coupled first order differential equations for the coefficient functions a_0, a_i, a_{ij}, a_{ijk}, $a_{ijkl}, ...$ etc. The mutually consistent solutions of these partial differential equations (PDEs) for potential V, give the invariant. As this method gives exact invariants for a system, one can utilize it to find higher order invariants for both real and complex Hamiltonian systems in two or higher dimensions. We will further elaborate the rationalization method for construction of higher order classical and quantum invariants of a number of systems.

In the present chapter, we will give self-contained presentation of the work i.e, the construction of second order classical invariants for a number of dynamical systems employing rationalization method in one and two dimensions in section 2 and 3. Section 4 deals with the construction of fourth order complex invariants using complex coordinates in ECPS.

2.2 Construction of second order invariants in one-dimension

In physics, phase space is a concept which unifies classical (Hamiltonian) mechanics and quantum mechanics. In classical mechanics, the phase space is the space of all possible states of a physical system. One needs both the position and momentum of system in order to determine the future behavior of that system. Consider one such one-dimensional real phase space (x, p, t), which can be transformed into a complex space (x_1, p_2, p_1, x_2, t), by defining position and momenta variables in ECPS as

$$x = x_1 + ip_2; \quad p = p_1 + ix_2. \tag{2.5}$$

Of course, the presence of variables (p_2, x_2) in the above transformation eq.(1.1), can be regarded as some sort of coordinate-momentum interaction of the dynamical system [4].

Thus for function I to be the TD (time dependent) dynamical invariant of the system in complex phase space, then this must conform the following invariance condition:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H] = 0. \tag{2.6}$$

where $[., .]$ is the Poisson bracket given by

$$[I, H] = \frac{\partial I}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial I}{\partial p}\frac{\partial H}{\partial x}. \tag{2.7}$$

The Hamiltonian $H(x, p, t)$ of a one-dimensional system in complex space can be expressed, using eq.(1.1), as

$$H = H_1(x_1, p_2, p_1, x_2, t) + iH_2(x_1, p_2, p_1, x_2, t). \tag{2.8}$$

Note that $(x_1, p_1), (x_2, p_2)$ constitute canonical pairs. Now consider a complex phase space function $I(x, p, t)$ as

$$I = I_1(x_1, p_2, p_1, x_2, t) + iI_2(x_1, p_2, p_1, x_2, t). \tag{2.9}$$

Thus for function I to be the TD (time-dependent) dynamical invariant of the system in complex phase space, then this must conform the following invariance condition: $\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H] = 0$. where $[.,.]$ is the Poisson bracket, which in view of the definition eq.(2.5), turns out to be

$$[I, H]_{(x,p)} = [I, H]_{(x_1,p_1)} - i[I, H]_{(x_1,x_2)} - i[I, H]_{(p_2,p_1)} - [I, H]_{(p_2,x_2)}. \tag{2.10}$$

Now using $I = I_1 + iI_2$, $H = H_1 + iH_2$ in eq.(2.7). Where I_1, I_2, H_1 and H_2 are real functions real variables (x_1, p_2, p_1, x_2, t), Eq.(2.6) will yield

$$\begin{aligned}&\frac{\partial}{\partial t}(I_1 + iI_2) + (\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_2})(I_1 + iI_2)(\frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_2})(H_1 + iH_2)\\ &-(\frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_2})(I_1 + iI_2)(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_2})(H_1 + iH_2) = 0.\end{aligned} \tag{2.11}$$

After equating real and imaginary parts separately to zero, one obtains the following pair of equations:

$$\begin{aligned}&\frac{\partial I_1}{\partial t} + (\frac{\partial I_1}{\partial x_1} + \frac{\partial I_2}{\partial p_2})(\frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_2}) - (\frac{\partial I_2}{\partial x_1} - \frac{\partial I_1}{\partial p_2})(\frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_2})\\ &-(\frac{\partial I_1}{\partial p_1} + \frac{\partial I_2}{\partial x_2})(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_2}) + (\frac{\partial I_2}{\partial p_1} - \frac{\partial I_1}{\partial x_2})(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_2}) = 0;\end{aligned} \tag{2.12}$$

And imaginary part is

$$\begin{aligned}&\frac{\partial I_2}{\partial t} + (\frac{\partial I_2}{\partial x_1} - \frac{\partial I_1}{\partial p_2})(\frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_2}) + (\frac{\partial I_1}{\partial x_1} + \frac{\partial I_2}{\partial p_2})(\frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_2})\\ &-(\frac{\partial I_2}{\partial p_1} - \frac{\partial I_1}{\partial x_2})(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_2}) - (\frac{\partial I_1}{\partial p_1} + \frac{\partial I_2}{\partial x_2})(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_2}) = 0.\end{aligned} \tag{2.13}$$

If Hamiltonian phase space does not involves the time t, then we can rewrite this and the term $\frac{\partial I_1}{\partial t}$ and $\frac{\partial I_2}{\partial t}$ in eq.(2.12) and eq.(2.13) must equal to zero. Now consider if imaginary parts I_2 and H_2 in I and H, are absent, then Eqn.(2.12) and (2.13) can rearrange this to get

$$\frac{\partial I_1}{\partial t} + (\frac{\partial I_1}{\partial x_1})(\frac{\partial H_1}{\partial p_1}) - (\frac{\partial I_1}{\partial p_2})(\frac{\partial H_1}{\partial x_2}) - (\frac{\partial I_1}{\partial p_1})(\frac{\partial H_1}{\partial x_1}) + (\frac{\partial I_1}{\partial x_2})(\frac{\partial H_1}{\partial p_2}) = 0. \tag{2.14}$$

and

$$\frac{\partial I_2}{\partial t} - (\frac{\partial I_1}{\partial p_2})(\frac{\partial H_1}{\partial p_1}) + (\frac{\partial I_1}{\partial x_1})(\frac{\partial H_1}{\partial x_2}) + (\frac{\partial I_1}{\partial x_2})(\frac{\partial H_1}{\partial x_1}) + (\frac{\partial I_1}{\partial p_1})(\frac{\partial H_1}{\partial p_2}) = 0. \tag{2.15}$$

The prescription for the construction of complex invariant I is same as that used earlier for the real Hamiltonian system. In brief, it can be listed as follows.

For a Hamiltonian phase space $H(x, p, t)$ we consider I (as an ansatz) in form of momentum p, using extended complex phase space approach both H and I can be set in the form $I = I_1 + iI_2$, $H = H_1 + iH_2$, and then put the I_1, I_2, H_1 and H_2 in Eqn. (2.12) and (2.13) with respect to power of p_1 and x_2 and the expression will yield coupled partial differential equations for the arbitrary complex coefficient functions appearing in the ansatz for I. The substitution of solutions of these partial equations (if the solutions are unique) in the ansatz for I, then this will yield the final form of invariant I. in forthcoming section we will discuss some examples from classical and quantum dynamical system

2.2.1 Some examples of dynamical system

There are various and different examples from classical and quantum dynamical system, but in present sections we will discuss few of them

1. Simple harmonic oscillator system

The theory of Simple harmonic oscillator is widely used in physics *viz*; black body radiation, kinetic theory of gases, electron in a box, diffraction in solids and so on. Oscillations of molecules can usually be analyzed fairly accurately as simple harmonic oscillations, in particular the diatomic molecule.

Our aim is to construct the complex invariant for SHO. For this we make the use of the methods discussed in previous section. Consider the case of simple harmonic oscillator systems with in the frame work of rationalization method.

Note that for simple harmonic oscillator systems

$$H = \frac{1}{2}p^2 + \frac{1}{2}w^2x^2. \tag{2.16}$$

one possible complex invariant for TD harmonic oscillator system, $u = ln(p + im\omega x) - i\omega t$ (as cited in [2]), has already been known in literature. Here we establish the complex version of (2.16), known as $\mathcal{PT}$-symmetric version is obtained by using (2.5) in (2.16), as $H = H_1 + iH_2$ with

$$H_1 = \frac{1}{2}(p_1^2 - x_2^2 + w^2x_1^2 - w^2p_2^2); \qquad H_2 = p_1x_2 + w^2x_1p_2. \tag{2.17}$$

With concerns to the physical insight into the complexified form of the Hamiltonian, the following observation is worth mentioning. In fact, it is found [9] that if H_1 can be identified with the (real) Hamiltonian of a two-dimensional physical system, then in several cases (studied in Ref. [9]) the analyticity property of H suggests that H_2 is the second integral of the motion of the system in the sense that $[H_1, H_2] = 0$, and H_1 and H_2 are linearly independent with respect to the canonical pairs $(x_1, p_1), (x_2, p_2), (x_3, p_3)$ and (x_4, p_4).

The above systems may allow a complex invariant I in the form

$$I = a_0(x) + a_2(x)p^2. \tag{2.18}$$

and write its complex translation in the form

$$I = I_1 + iI_2. \tag{2.19}$$

where

$$I_1 = a_{0r} + a_{2r}(p_1^2 - x_2^2) - a_{2i}p_1x_2; \qquad I_2 = a_{0i} + a_{2i}(p_1^2 - x_2^2) + a_{2r}p_1x_2. \tag{2.20}$$

and the complex coefficient functions $a_0(x)$, and $a_2(x)$ are written in the form

$$a_0(x) = a_{0r}(x_1, p_2) + ia_{0i}(x_1, p_2); \qquad a_2(x) = a_{2r}(x_1, p_2) + ia_{2i}(x_1, p_2). \tag{2.21}$$

with a_{0r}, a_{0i}, a_{2r} and a_{2i} are the real functions of their real arguments. Substitution of (2.17), (2.18) in (2.12) yields the expression

$$\Big[(\frac{\partial a_{0r}}{\partial x_1} + \frac{\partial a_{0i}}{\partial p_2}) + (\frac{\partial a_{2r}}{\partial x_1} + \frac{\partial a_{2i}}{\partial p_2})(p_1^2 - x_2^2) + (2p_1x_2)(\frac{\partial a_{2r}}{\partial p_2} - \frac{\partial a_{2i}}{\partial x_1})\Big](2p_1)$$

$$-\Big[(\frac{\partial a_{0i}}{\partial x_1} - \frac{\partial a_{0r}}{\partial p_2}) + (\frac{\partial a_{2i}}{\partial x_1} + \frac{\partial a_{2r}}{\partial p_2})(p_1^2 - x_2^2) + (2p_1x_2)(\frac{\partial a_{2r}}{\partial x_1} + \frac{\partial a_{2i}}{\partial p_2})\Big](2x_2)]$$

$$-(p_1a_{2r} - x_2a_{2i})(8w^2x_1) + (p_1a_{2i} + x_2a_{2r})(8w^2p_2) = 0. \tag{2.22}$$

which can be rationalized with respect to the power of x_1, p_2 and their combinations to give the following set of four coupled partial differential equations

$$\frac{\partial a_{0r}}{\partial x_1} + \frac{\partial a_{0i}}{\partial p_2} - 4w^2x_1a_{2r} + 4w^2p_2a_{2i} = 0;; \tag{2.23}$$

$$\frac{\partial a_{0r}}{\partial p_2} + \frac{\partial a_{0i}}{\partial x_1} + 4w^2x_1a_{2i} + 4w^2p_2a_{2r} = 0;; \tag{2.24}$$

$$\frac{\partial a_{2i}}{\partial x_1} - \frac{\partial a_{2r}}{\partial p_2} = 0;; \qquad \frac{\partial a_{2r}}{\partial x_1} + \frac{\partial a_{2i}}{\partial p_2} = 0.. \tag{2.25}$$

So for construction of complex invariants in one-dimensions one has to find out solutions for following unknown parameters a_{0r}, a_{0i}, a_{2r} and a_{2i} which are functions of (x_1, p_2).

(i) Solutions for a_{2r}, a_{2i}.

For determination of a_{2r}, a_{2i}, equations (2.25) can be reduced to similar second-order forms for the functions a_{2r}, a_{2i}, respectively,as

$$\frac{\partial^2 a_{2r}}{\partial x_1^2} + \frac{\partial^2 a_{2r}}{\partial p_2^2} = 0; \qquad \frac{\partial^2 a_{2i}}{\partial x_1^2} + \frac{\partial^2 a_{2i}}{\partial p_2^2} = 0. \tag{2.26}$$

Assuming separability of a_{2r} and a_{2i} under addition as $a_{2r} = X_{2r}(x_1) + P_{2r}(p_2), a_{2i} = X_{2i}(x_1) + P_{2i}(p_2)$ it is not difficult to obtain the solution of (2.26) in the form

$$\begin{aligned} a_{2r} &= \frac{\alpha}{2}(x_1^2 - p_2^2) + \alpha_1 x_1 + \alpha_2 p_2 + \delta_1; \\ a_{2i} &= \frac{\beta}{2}(x_1^2 - p_2^2) + \beta_1 x_1 + \beta_2 p_2 + \delta_2. \end{aligned} \tag{2.27}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2$ are arbitrary constant of integration to be determined later.

(ii) Solutions for a_{0r}, a_{0i}

Differentiating (2.25) with respect to x_1 and again (2.25) with respect to p_2 and add

$$\begin{aligned} \frac{\partial^2 a_{0r}}{\partial x_1^2} + \frac{\partial^2 a_{0r}}{\partial p_2^2} &= 4w^2 x_1 (\frac{\partial a_{2r}}{\partial x_1} - \frac{\partial a_{2i}}{\partial p_2}) - 4w^2 p_2 (\frac{\partial a_{2i}}{\partial x_1} + \frac{\partial a_{2r}}{\partial p_2}) \\ &= -8w^2 (x_1 \frac{\partial a_{2i}}{\partial p_2} + p_2 \frac{\partial a_{2i}}{\partial x_1}) \\ &= -8w^2 (\beta_2 x_1 - \beta_1 p_2). \end{aligned} \tag{2.28}$$

where we have used Eqs.(2.25) and then expression (2.27) to simplify the right hand side. For solution of (2.28) we again assume a separable form $a_{0r}(x_1, p_2) = X_{0r}(x_1) + P_{0r}(p_2), a_{0i} = X_{0i}(x_1) + P_{0i}(p_2)$ this lead to pair of ordinary partial differential equations whose solution immediately will yield

$$a_{0r}(x_1, p_2) = -\frac{4}{3} w^2 (\beta_2 x_1^3 + \beta_1 p_2^3) + \frac{c_0}{2}(x_1^2 - p_2^2) + c_1 x_1 + c_2 p_2 + c_3. \tag{2.29}$$

where c_o, c_1, c_2 and c_3 are the arbitrary constants of integration. Another result for a_{0r} can be obtained if one retains the term $(\frac{\partial a_{2r}}{\partial p_2})$ and $(\frac{\partial a_{2r}}{\partial x_1})$ in (2.29) instead of $(\frac{\partial a_{2i}}{\partial p_2})$ and $(\frac{\partial a_{2i}}{\partial x_1})$ and looks for the solution of the resultant partial differential equations again in the separable in the form x_1 and p_2. The corresponding result for a_{0r} then becomes

$$a_{0r}(x_1, p_2) = \frac{2}{3} w^2 \nu (x_1^4 + p_2^4) + \frac{4}{3} w^2 (\alpha_1 x_1^3 - \alpha_2 p_2^3) + \frac{\bar{c_0}}{2}(x_1^2 - p_2^2) + \bar{c_1} x_1 + \bar{c_2} p_2 + \bar{c_3}. \tag{2.30}$$

Uniqueness of expression (2.29) and (2.30) for a_{0r} would require that

$$\alpha = 0, \beta_2 = -\alpha_1, \alpha_1 = \beta_2, c_0 = \bar{c_0}, c_1 = \bar{c_1}, c_2 = \bar{c_2}, c_3 = \bar{c_3}. \tag{2.31}$$

This will yield acceptable form of a_{0r} as

$$a_{0r}(x_1, p_3) = -\frac{4}{3} w^2 (\rho_2 x_1^3 + \rho_1 p_3^3) + c_0 (x_1^2 - p_3^2) + c_1 x_1 + c_2 p_3 + c_3. \tag{2.32}$$

For determination of a_{0i} one follows the same procedure as followed for a_{0r} and obtains the coefficient function $a_{0r}(x_1, p_2)$ in the form

$$a_{0i}(x_1, p_2) = \frac{4}{3} w^2 (\beta_1 x_1^3 - \beta_2 p_2^3) + d_0 (x_1^2 - p_2^2) + d_1 x_1 + d_2 p_2 + d_3 \tag{2.33}$$

where d_o, d_1, d_2 and d_3 are the arbitrary constants of integration. With the choice (2.30)(including now $\alpha = 0$)for the arbitrary constants the expressions a_{2r} and a_{2i} now takes the form

$$a_{2r} = -\beta_2 x_1 + \beta_1 p_2 + \delta_1; \qquad a_{2i} = \beta_1 x_1 + \beta_2 p_2 + \delta_2 \tag{2.34}$$

Note that the forms [(2.31)-(2.34)] of $a_{0r}, a_{0i}, a_{2r}, a_{2i}$ are determined only from (2.14). With this expressions for the coefficient function when (2.15) is rationalized, one obtained several constrains relations among the arbitrary constants appearing in Eqn. [(2.31)-(2.34)], thereby reducing the number of arbitrary constants in the final solutions. The constrained so obtained are $c_1 = -d_2, \qquad d_1 = c_2, \qquad c_0 = 4w^2\delta_1, \qquad d_0 = 4w^2\delta_2.$
which gives rise to forms of coefficient functions as

$$a_{2r} = -\beta_2 x_1 + \beta_1 p_2 + \frac{c_0}{4w^2}; \qquad a_{2i} = \beta_1 x_1 + \beta_2 p_2 + \frac{d_0}{4w^2};$$

$$\begin{aligned} a_{0r}(x_1, p_2) &= -\frac{4}{3}w^2(\beta_2 x_1^3 + \beta_1 p_2^3) + c_0(x_1^2 - p_2^2) + c_1 x_1 + d_1 p_2 + c_3; \\ a_{0i}(x_1, p_2) &= \frac{4}{3}w^2(\beta_1 x_1^3 - \beta_2 p_2^3) + d_0(x_1^2 - p_2^2) + d_1 x_1 - c_1 p_2 + d_3. \end{aligned} \tag{2.35}$$

Construction of Invariants. For the construction of complex invariants using the results (2.35) for $a_{0r}, a_{0i}, a_{2r}, a_{2i}$ one can obtain the complex invariant I from (2.20), in which the real and imaginary parts I_1 and I_2 are given by

$$\begin{aligned} I_1 &= \frac{4}{3}w^2(\beta_2 x_1^3 + \beta_1 p_2^3) + \frac{c_0}{2}(x_1^2 - p_2^2) + c_1 x_1 + d_1 p_2 \\ &+ (p_1^2 - x_2^2)(-\beta_2 x_1 + \beta_1 p_2 + \frac{c_0}{2w^2}) - 2p_1 x_2(\beta_1 x_1 + \beta_2 p_2 + \frac{d_0}{2w^2}); \end{aligned} \tag{2.36}$$

$$\begin{aligned} I_2 &= \frac{4}{3}w^2(\beta_1 x_1^3 - \beta_2 p_2^3) + \frac{d_0}{2}(x_1^2 - p_2^2) + d_1 x_1 - c_1 p_2 \\ &+ (p_1^2 - x_2^2)(\beta_1 x_1 + \beta_2 p_2 + \frac{d_0}{2w^2}) + 2p_1 x_2(-\beta_2 x_1 + \beta_1 p_2 + \frac{c_0}{2w^2}). \end{aligned} \tag{2.37}$$

Finally the complex invariant $I = I_1 + iI_2$ can be written as

$$I = \frac{4}{3}w^2 b(x_1^3 + ip_2^3) + \frac{\sigma_0}{2}(x_1^2 - p_2^2) + \sigma_1(x_1 - ip_2) + (p_1^2 - x_2^2 + 2ip_1x_2)(bx_1 - ibp_2 + \frac{\sigma_0}{4w^2}) \tag{2.38}$$

or on simplification we can write

$$I = \frac{4}{3}w^2 b(x_1^3 + ip_2^3) + \frac{\sigma_0}{2}(x_1^2 - p_2^2) + \sigma_1(x_1 - ip_2) + (p_1^2 - x_2^2 + 2ip_1x_2)(bx_1 - ibp_2 + \frac{\sigma_0}{4w^2}). \tag{2.39}$$

where $b = -\beta_2 + i\beta_1; \quad \sigma_0 = c_0 + id_0; \quad \sigma_1 = c_1 + id_1$ are the arbitrary constants. In real and complex coordinate it is written as

$$I = \frac{b\omega^2}{3}x^*(3x^2 + x^{*2}) + \frac{\sigma_0}{4}(x^2 + x^{*2}) + \sigma_1 x^* + (bx^* + \frac{\sigma_0}{4\omega^2})p^2$$

where $x^* = x_1 - ip_2, \; p^* = p_1 - ix_2.$ are as defined earlier, which conforms the invariance condition for integrability with view of poisson bracket.

2. Non-hermitian $\mathcal{PT}$-symmetric Hamiltonian

This type of Non-hermitian $\mathcal{PT}$-symmetric Hamiltonian have been discussed by Bender *et al* [1]. With a motivation for constructing complex invariant for above systems we make the use of methods discussed in previous section. Note that for Non-$\mathcal{PT}$-symmetric Hamiltinian systems

$$H = \frac{1}{2}p^2 + ix + ix^3 \tag{2.40}$$

The complex version of (2.40), i.e. the $\mathcal{PT}$-symmetric form can acquire by using (2.5) in (2.40), as $H = H_1 + iH_2$ with

$$H_1 = \frac{1}{2}(p_1^2 - x_2^2) + p_2^3 - 3x_1^2p_2 - p_2); \quad H_2 = p_1x_2 + x_1^3 - 3x_1p_2^2 + x_1. \tag{2.41}$$

and we make an assumption that it posses a complex invariant I in the form

$$I = a_0(x) + a_2(x)p^2.$$

this can be rewritten in its complex variant, in the form $I = I_1 + iI_2$. where

$$I_1 = a_{0r} + a_{2r}(p_1^2 - x_2^2) - a_{2i}p_1x_2; \quad I_2 = a_{0i} + a_{2i}(p_1^2 - x_2^2) + a_{2r}p_1x_2. \tag{2.42}$$

Substitution of (2.41), (2.42) in (2.12) yields the expression and rationalization of the resultant expressions with respect to the powers of p_1, x_2 and their combinations gives a set of four coupled pdes. These pdes can again be solved by following the procedure adopted in the previous case. For the construction of complex invariants using the results from solutions of pdes for $a_{0r}, a_{0i}, a_{2r}, a_{2i}$ one can obtain the complex invariant I from (2.20), in which the real and imaginary parts I_1 and I_2 are given by

$$I_1 = 2\beta_2(x_1^3p_2 + x_1p_2^3) - \beta_1(x_1^4 - p_2^4) - \beta_1(x_1^2 + p_2^2) + \frac{1}{2}(p_1^2 - x_2^2)(-\beta_2x_1 + \beta_1p_2)$$
$$-p_1x_2(\beta_1x_1 + \beta_2p_2). \tag{2.43}$$

$$I_2 = -2\beta_1(x_1^3p_2 + x_1p_2^3) - \beta_2(x_1^4 - p_2^4) - \beta_2(x_1^2 + p_2^2) + \frac{1}{2}(p_1^2 - x_2^2)(\beta_1x_1 + \beta_2p_2)$$
$$+p_1x_2(-\beta_2x_1 + \beta_1p_2). \tag{2.44}$$

Finally the complex invariant $I = I_1 + iI_2$ can be written as

$$I = 2b(x_1^3p_2 + x_1p_2^3) - (x_1^4 - p_2^4) - (x_1^2 + p_2^2) + \sigma_1(x_1 - ip_2) + (p_1^2 - x_2^2 + 2ip_1x_2)(bx_1 - ibp_2). \tag{2.45}$$

or on simplification we can write

$$I = ibx^3x^* + ibxx^* + \frac{1}{2}bx^*p^2. \tag{2.46}$$

3. $\mathcal{PT}$-symmetric Hamiltonian system

Note that for $\mathcal{PT}$-symmetric Hamiltonian system

$$H = \frac{1}{2}p^2 + x + ix^3. \tag{2.47}$$

Here we demonstrate that the complex version of (2.47), namely the $\mathcal{PT}$-symmetric one obtained by using (2.5) in (2.47), as $H = H_1 + iH_2$ with

$$H_1 = \frac{1}{2}(p_1^2 - x_2^2) + x_1 - 3x_1^2 p_2 + p_2^3; \quad H_2 = p_1 x_2 + x_1^3 - 3x_1 p_2^2 + p_2. \tag{2.48}$$

Following the prescription outlined in previous example the coefficient functions comes out to be

$$\begin{aligned} a_{0r}(x_1, p_2) &= 2\beta_2(x_1^3 p_2 + x_1 p_2^3) - \beta_1(x_1^4 - p_2^4) - \beta_2(x_1^2 + p_2^2). \\ a_{0i}(x_1, p_2) &= -2\beta_1(x_1^3 p_2 + x_1 p_2^3) - \beta_2(x_1^4 - p_2^4) + \beta_1(x_1^2 + p_2^2). \end{aligned} \tag{2.49}$$

Using the results for the coefficient functions, the complex invariant I can be written as

$$I = ibx^3 x^* + bxx^* + \frac{1}{2}bx^* p^2 \tag{2.50}$$

which conforms the invariance condition for integrability with view of poisson bracket.

4. Complex cubic potential

For constructing complex invariant for complex cubic potential here, we first consider the case of $\mathcal{PT}$-symmetric Hamiltonians systems. Spectral analysis of the complex cubic oscillator has been carried out by many researchers [8]. Using the exact semiclassical analysis, They studied the spectrum of a one-parameter family of complex cubic oscillators. Note that Hamiltonian for complex cubic potential is

$$H = p^2 + \delta_1(ix) + \delta_2(ix)^2 + \delta_3(ix)^3. \tag{2.51}$$

The complex version of (2.51), i.e. the $\mathcal{PT}$-symmetric version obtained by using (2.5) in (2.51), as $H = H_1 + iH_2$ with

$$\begin{aligned} H_1 &= p_1^2 - x_2^2 - \delta_1 p_2 - \delta_2(x_1^2 - p_2^2) - \delta_3 p_2^3 + 3\delta_3 x_1^2 p_2; \\ H_2 &= 2p_1 x_2 + \delta_1 x_1 - 2\delta_2 x_1 p_2 - \delta_3 x_1^3 + 3\delta_3 x_1 p_3^2. \end{aligned} \tag{2.52}$$

Following the same procedure for the coefficient functions, one can obtain the complex invariant

$$I = ib[x^* x\delta_1 - \frac{\delta_2}{3}x^*(x^{*2} - 3x^2) - i\delta_3 x^* x^3 + x^* p^2].$$

conforms the invariance condition for integrability.

5. Integral corresponding to non-linear evolution equations

The Koerteweg de-Vries (KdV) equations has wide ranging applications in non-linear physics [45]. More physical insight into mathematical content of [45] can be employed searching an invariant for a Hamiltonian derived from this equation. The KdV equations in one-dimensional real field $u(x,t)$ is given by

$$\frac{\partial u}{\partial t} + \bar{a}u\left(\frac{\partial u}{\partial x}\right) + \eta\left(\frac{\partial^3 u}{\partial^3 x}\right). \tag{2.53}$$

where $\bar{a}$ and η are constant. In stationary frame $\xi = (x - \gamma t)$, Eq(2.53) becomes

$$\eta \frac{\partial^2 u}{\partial^2 \xi} = \gamma u - \frac{1}{2}\bar{a}u^2.$$

Where γ is a free parameter characterizing the velocity of a stationary wave. Make an identification, such as u as the position coordinate and ξ as time coordinate and η as mass parameter, the above equation becomes identical with equation of motion for one-dimensional anharmonic oscillator described by the potential,

$$V(u) = -\left(\frac{\gamma}{2}\right)u^2 + \left(\frac{\bar{a}}{6}\right)u^3. \tag{2.54}$$

Hamiltonian correlated to this potential, can be written as

$$H = \frac{1}{2}p^2 - \frac{\gamma}{2}x^2 + \left(\frac{\bar{a}}{6}\right)x^3. \tag{2.55}$$

using (2.5) in (2.55), gives us $H = H_1 + iH_2$ with

$$\begin{aligned} H_1 &= \frac{1}{2}(p_1^2 - x_2^2) - \frac{\gamma}{2}(x_1^2 - p_2^2) + \frac{\bar{a}}{6}(x_1^3 - 3x_1p_2^2); \\ H_2 &= p_1x_2 - \gamma x_1 p_2 + \frac{\bar{a}}{6}(3x_1^2p_2 - p_2^3). \end{aligned} \tag{2.56}$$

Substitution of (2.56), in (2.12) yields the expression and rationalization of the resultant expressions gives solutions for a_{2r}, a_{2i}

$$\begin{aligned} a_{0r} &= \beta_2\gamma x_1p_2^2 + \beta_1\gamma x_1^2p_2 + \frac{1}{3}\beta_1\gamma p_2^3 + \frac{1}{3}\beta_2\gamma x_1^3 - \frac{\bar{a}}{6}\beta_2(x_1^4 - p_2^4) - \frac{\bar{a}}{3}\beta_1 x_1p_2^3 + x_1^3p_2; \\ a_{0i} &= -\beta_1\gamma x_1p_2^2 + \beta_2\gamma x_1^2p_2 + \frac{1}{3}\beta_2\gamma p_2^3 - \frac{1}{3}\beta_1\gamma x_1^3 + \frac{\bar{a}}{6}\beta_1(x_1^4 - p_2^4) - \frac{\bar{a}}{3}\beta_2(x_1p_2^3 + x_1^3p_2). \end{aligned} \tag{2.57}$$

one can obtain the complex invariant I from

$$I = -\frac{b\gamma}{6}x^*(3x^2 - x^2) + \frac{b\bar{a}}{6}x^3x^* + \frac{b}{2}x^*p^2 \tag{2.58}$$

where $b = -\beta_2 + i\beta_1$; are the arbitrary constants.

6. Shifted harmonic oscillator complex x-plane

This type of oscillators has been discussed in context of neutron scattering [10]. They have been used to study the spin-orbit coupling term in shell model by neutron scattering using complex harmonic oscillator potential. Recalling that for shifted oscillator complex x-plane

$$H = \frac{1}{2}p^2 + \frac{1}{2}w^2(x + \frac{1}{2}i\gamma)^2. \tag{2.59}$$

where ω and γ are real constants. This form of H, after appropriate scaling of x and p (with $\omega = 1$) expressing it as,

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + i\gamma x.$$

using (2.5) in (2.59), gives as $H = H_1 + iH_2$ with

$$H_1 = \frac{1}{2}(p_1^2 - p_2^2 + x_1^2 - x_2^2) - \gamma p_2; \quad H_2 = p_1 x_2 + x_1 p_2 + \gamma x_1.$$

co-efficient functions a_{0r}, a_{0i} appears to be

$$a_{0r}(x_1, p_2) = -\frac{4}{3}(\beta_2 x_1^3 + \beta_1 p_2^3) - \beta_1\gamma(x_1^2 + p_2^2) + c_1 x_1 + d_1 p_2 + c_3;$$
$$a_{0i}(x_1, p_2) = \frac{4}{3}(\beta_1 x_1^3 - \beta_2 p_2^3) - \beta_2\gamma(x_1^2 + p_2^2) + d_1 x_1 - c_1 p_2 + d_3. \tag{2.60}$$

one can find the complex invariant I in the from $I = I_1 + iI_2$ as

$$I = \frac{b}{3}x^*(3x^2 + x^2) + ib\gamma x x^* + \sigma_1 x^* + bx^* p^2. \tag{2.61}$$

which conforms the invariance condition for integrability.

7. Quartic potential

The advent of high-speed digital computers creates revolution numerical calculations. this leads to considerable interest in highly anharmonic vibrational systems. The simplest of these systems is the one-dimensional quartic oscillator. To date, the most comprehensive results are obtained by many researchers, whilst the most accurate energy levels have been calculated by Secrest *et al* [11]. This has indeed been found to be the case for several molecules, for example, trimethylene oxide, cyclobutane, and cyclobutanone. Invariants for this type of system will useful in understanding the complex trajectories. For constructing complex invariant for complex quartic potential here, consider the case of complex quartic potential systems. Hamiltonian complex quartic potential systems

$$H = \frac{1}{2}p^2 + \frac{1}{2}\delta_1^2 x^2 + \frac{1}{4}\delta_2 x^4 \tag{2.62}$$

using (2.5) in (2.62) allow us to write

$$H_1 = \frac{1}{2}(p_1^2 - x_2^2) + \frac{1}{2}\delta_1(x_1^2 - p_2^2) + \frac{1}{4}\delta_2(x_1^4 + p_2^4 - 6x_1^2 p_2^2);$$
$$H_2 = p_1 x_2 + \delta_1 x_1 p_2 + \delta_2(x_1^3 p_2 - x_1 p_2^3).$$

This concedes coefficients

$$a_{0r} = -\frac{1}{3}\delta_1(\beta_2 x_1^3 + \beta_1 p_3^3) - \delta_1(\beta_2 x_1 p_2^2 + \beta_1 x_1^2 p_2) + \delta_2(\beta_2 x_1 p_2^4 - \beta_1 x_1^4 p_2) + \frac{1}{5}\delta_2(\beta_1 p_2^5 - \beta_2 x_1^5);$$
$$a_{0i} = \frac{1}{3}\delta_1(\beta_1 x_1^3 - \beta_2 p_2^3) + \delta_1(\beta_1 x_1 p_2^2 - \beta_2 x_1^2 p_2) - \delta_2(\beta_1 x_1 p_2^4 + \beta_2 x_1^4 p_2) + \frac{1}{5}\delta_2(\beta_2 p_2^5 + \beta_1 x_1^5). \tag{2.63}$$

Using the results (2.63), Invariant can be written as

$$I = \frac{b\delta_1}{6}x^*(3x^2 - x^2) + \frac{b\delta_2}{20}(5x^4 x^* - x^5) + \frac{b}{2}x^* p^2. \tag{2.64}$$

which conforms the invariance condition for integrability.

2.3 Construction of second order invariants in two-dimension

In classical dynamics a TID Hamiltonian system with two degrees of freedom is integrable if a constant of motion is known, I, independent of the hamiltonian $H, [I, H]_{PB} = 0, [,]_{PB}$ is the Poisson bracket. Any trajectory of the system in a four-dimensional phase space lies on one of the hypersurfaces [H = const., I = const.] which should be two-dimensional. The integrability of classical systems with a hamiltonian

$$H = \sum_{i=1}^{2} \frac{1}{2} p_i^2 + V(p_i).$$

is established in some cases of interest for physics [16]. Recently it has been shown that the study of nonlinear systems of this type may be useful for solving the problem of integrability of the Yang-Mills equations in classical field theory [1-2].

Colt [24] in his article constructed Invariant for two dimensional TID Hamiltonian systems namely

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \epsilon\left[(x^2+\delta)y^{-\frac{2}{3}} + \frac{3}{4}y^{\frac{4}{3}}\right] \tag{2.65}$$

and its invariants found to be

$$I = 2p_x^3 + 3p_x p_y^2 + 3\epsilon p_x\left[(x^2+\delta)y^{-\frac{2}{3}} - 3y^{\frac{4}{3}} + 18p_y \epsilon x y^{\frac{1}{3}}\right] \tag{2.66}$$

The second type of Hamiltonian taken by Colt is

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + x^2 + 4y^2 + \delta^{-2} \tag{2.67}$$

and its invariants

$$I = p_x^2 p_y + 2\epsilon(4xyp_x - p_y x^2 + p_y \delta x^{-2}) \tag{2.68}$$

Inozemtsev [12] shown that a classical system with hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \lambda(xy)^{-\frac{2}{3}}.$$

possesses a constant of the motion

$$I = p_x p_y(p_x y - p_y x) + 2\lambda(yp_y - xp_x)(xy)^{-\frac{2}{3}}.$$

One can clearly note that in all the above cases

$$[H, I\ ] = 0 \tag{2.69}$$

but the quantum system with the same hamiltonian has no constants of the motion which are polynomials in the momenta of order not higher than three, except H.

Feng [13] in his letter, constructed integrals for a two-dimensional Hamiltonian system with a quartic potential, [11] whose Hamiltonian can be expressed as

$$H = \frac{1}{2}(p_x^2 + p_y^2) + Ax^2 + By^2 + \alpha x^4 + \beta y^4 + \delta x^2 y^2.$$

and their, the equations of motion are given by

$$\dot{x} = p_x; \qquad \dot{p}_x = 2Ax - 2x(2\alpha x^2 + \delta y^2);$$
$$\dot{y} = p_y; \qquad \dot{p}_y = -2By - 2y(2\beta y^2 + \delta x^2).$$

or

$$\ddot{x} = -2Ax - 2x(2\alpha x^2 + \delta y^2);$$
$$\ddot{y} = -2\beta y - 2y(2\beta y^2 + \delta x^2).$$

The above Hamiltonian system is known to be integrable for the following four cases [17,18,19]:

(I) $\beta = \alpha, \delta = 2\alpha,$ α, A, B arbitrary,

(II) $\beta = \alpha, \delta = 6\alpha, A = B,$ α, A arbitrary,

(III)$\beta = 16\alpha, \delta = 12\alpha, B = 4A,$ α, A arbitrary,

(IV) $\beta = 8\alpha, \delta = 6\alpha, B = 4A,$ α, A arbitrary.

With the corresponding second integrals listed as follows:

$$(I) \qquad I_1 = I(x, y, p_x, p_y) = (xp_y - yp_x)^2 + \frac{B - A}{\alpha}[p_x^2 + 2y^2\{A + \alpha(x^2 + y^2)\}];$$
$$(II) \qquad I_2 = I(x, y, p_x, p_y) = p_x p_y + 2xy[A + 2\alpha(x^2 + y^2)];$$
$$(III) \qquad I_3 = I(x, y, p_x, p_y) = (xp_y - yp_x)p_x + 2x^2 y\{A + \alpha(x^2 + y^2)\};$$
$$(IV) \qquad I_4 = I(x, y, p_x, p_y) = 4\alpha x^2(xp_y - 2yp_x)^2 + [p_x^2 + 2x^2\{A + \alpha(x^2 + y^2)\}]. \qquad (2.70)$$

So from above cases it is interesting to study the integrability of Dynamical systems in two dimensions. Here, in the present work we carry out the ECPS approach to obtain exact complex integrals of a two-dimensional classical dynamical system [4, 29]. Rationalization method method is explored for such constructions as this has been widely used in literature for the construction of exact real and complex integrals.

Kaushal *et al* [29] has investigated the complex invariants of one-dimensional complex Hamiltonian systems on the extended complex phase plane, characterized by $x = x_1 + ip_2$ and $p = p_1 + ix_2$. In this approach both x and p are separately made complex by extending each of them to the corresponding complex planes, i.e. inserting an imaginary component in each. From physics point of view p_2 and x_2 can be regarded as fictitious/spurious components of momentum and coordinate respectively and their presence in the above transformation equations as such allow the introduction of some sort of coordinate-momentum coupling of the dynamical system. However on the classical level only a few studies of one-dimensional systems have been reported in recent past.

In this section we carry out the extended phase plane approach to obtain exact complex invariants of a two-dimensional classical dynamical system. Mishra *et al* [30] have used the extended phase plane approach to obtain exact complex invariants of a two-dimensional classical dynamical system. The two-dimensional system on extended phase plane characterized by $x = x_1+ip_3;\ \ y = x_2+ip_4;\ \ p_x = p_1+ix_3;\ \ p_y = p_2+ix_4$. In this approach both x, y and p_x, p_y are separately made complex by extending each of them to the corresponding complex planes, i.e. inserting an imaginary component in each.

2.3.1 Formalism

The physical state of a classical and quantum dynamical system is completely described by the position vectors and momentum of the phase space. In Hamiltonian mechanics generalized coordinates and momenta are used instead$x = (x_1, x_2, ..., x_n);\ p = (p_1, p_2, ..., p_n)$.

In similar context consider a two-dimensional real phase space (x, y, p_x, p_y, t), and the transformation equations defining the position and momenta variables as

$$x = x_1 + ip_3; \qquad y = x_2 + ip_4; \qquad p_x = p_1 + ix_3; \qquad p_y = p_2 + ix_4. \tag{2.71}$$

the new variables (x_3, x_4, p_3, p_4) in the above transformation eq.(2.71), can be considered as a type of coordinate-momentum interaction of the dynamical system. The Hamiltonian $H(x, y, p_x, p_y, t)$ of a two-dimensional system in complex space can be expressed, using eq.(2.71), as

$$H = H_1(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t) + iH_2(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t). \tag{2.72}$$

We can rewrite eq.(2.71)

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_3}; \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4}; \quad \frac{\partial}{\partial p_x} = \frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3}; \quad \frac{\partial}{\partial p_y} = \frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}. \tag{2.73}$$

As we have already discussed that the invariant is also a complex phase space function $I(x, y, p_x, p_y, t)$, so consider it as

$$I = I_1(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t) + iI_2(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t). \tag{2.74}$$

For making sense of the Liouville integrability for function I, to be the TD (time-dependent) dynamical invariant of the system, then this must follow invariance condition:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H] = 0. \tag{2.75}$$

where $[., .]$ is the PB, which in view of the definition, eq.(2.72), turns out to be

$$\begin{aligned} [I, H]_{(x,p)} = {} & [I, H]_{(x_1,p_1)} - i[I, H]_{(x_1,x_3)} - i[I, H]_{(p_3,p_1)} - [I, H]_{(p_3,x_3)} \\ & + [I, H]_{(x_2,p_2)} - i[I, H]_{(x_2,x_4)} - i[I, H]_{(p_4,p_2)} - [I, H]_{(p_4,x_4)}. \end{aligned} \tag{2.76}$$

We can rewrite eq.(2.75) using $I = I_1 + iI_2$, $H = H_1 + iH_2$, where I_1, I_2, H_1 and H_2 are real functions real variables $(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t)$, Eq.(2.75) will yield

$$\frac{\partial}{\partial t}(I_1+iI_2)+(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial p_3})(I_1+iI_2)(\frac{\partial}{\partial p_1}-i\frac{\partial}{\partial x_3})(H_1+iH_2)-(\frac{\partial}{\partial p_1}-i\frac{\partial}{\partial x_3})$$
$$(I_1+iI_2)(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial p_3})(H_1+iH_2)+(\frac{\partial}{\partial x_2}-i\frac{\partial}{\partial p_4})(I_1+iI_2)(\frac{\partial}{\partial p_2}$$
$$-i\frac{\partial}{\partial x_4})(H_1+iH_2)-(\frac{\partial}{\partial p_2}-i\frac{\partial}{\partial x_4})(I_1+iI_2)(\frac{\partial}{\partial x_2}-i\frac{\partial}{\partial p_4})(H_1+iH_2)=0. \quad (2.77)$$

Separating and after equating the real and imaginary parts to zero, the following pair of equations are obtained: as real part is

$$\frac{\partial I_1}{\partial t}+(\frac{\partial I_1}{\partial x_1}+\frac{\partial I_2}{\partial p_3})(\frac{\partial H_1}{\partial p_1}+\frac{\partial H_2}{\partial x_3})-(\frac{\partial I_2}{\partial x_1}-\frac{\partial I_1}{\partial p_3})(\frac{\partial H_2}{\partial p_1}-\frac{\partial H_1}{\partial x_3})$$
$$-(\frac{\partial I_1}{\partial p_1}+\frac{\partial I_2}{\partial x_3})(\frac{\partial H_1}{\partial x_1}+\frac{\partial H_2}{\partial p_3})+(\frac{\partial I_2}{\partial p_1}-\frac{\partial I_1}{\partial x_3})(\frac{\partial H_2}{\partial x_1}-\frac{\partial H_1}{\partial p_3})$$
$$+(\frac{\partial I_1}{\partial x_2}+\frac{\partial I_2}{\partial p_4})(\frac{\partial H_1}{\partial p_2}+\frac{\partial H_2}{\partial x_4})-(\frac{\partial I_2}{\partial x_2}-\frac{\partial I_1}{\partial p_4})(\frac{\partial H_2}{\partial p_2}-\frac{\partial H_1}{\partial x_4})$$
$$-(\frac{\partial I_1}{\partial p_2}+\frac{\partial I_2}{\partial x_4})(\frac{\partial H_1}{\partial x_2}+\frac{\partial H_2}{\partial p_4})+(\frac{\partial I_2}{\partial p_2}-\frac{\partial I_1}{\partial x_4})(\frac{\partial H_2}{\partial x_2}-\frac{\partial H_1}{\partial p_4})=0; \quad (2.78)$$

and imaginary part is

$$\frac{\partial I_2}{\partial t}+(\frac{\partial I_2}{\partial x_1}-\frac{\partial I_1}{\partial p_3})(\frac{\partial H_1}{\partial p_1}+\frac{\partial H_2}{\partial x_3})+(\frac{\partial I_1}{\partial x_1}+\frac{\partial I_2}{\partial p_3})(\frac{\partial H_2}{\partial p_1}-\frac{\partial H_1}{\partial x_3})$$
$$-(\frac{\partial I_2}{\partial p_1}-\frac{\partial I_1}{\partial x_3})(\frac{\partial H_1}{\partial x_1}+\frac{\partial H_2}{\partial p_3})-(\frac{\partial I_1}{\partial p_1}+\frac{\partial I_2}{\partial x_3})(\frac{\partial H_2}{\partial x_1}-\frac{\partial H_1}{\partial p_3})$$
$$+(\frac{\partial I_2}{\partial x_2}-\frac{\partial I_1}{\partial p_4})(\frac{\partial H_1}{\partial p_2}+\frac{\partial H_2}{\partial x_4})+(\frac{\partial I_1}{\partial x_2}+\frac{\partial I_2}{\partial p_4})(\frac{\partial H_2}{\partial p_2}-\frac{\partial H_1}{\partial x_4})$$
$$-(\frac{\partial I_2}{\partial p_2}-\frac{\partial I_1}{\partial x_4})(\frac{\partial H_1}{\partial x_2}+\frac{\partial H_2}{\partial p_4})-(\frac{\partial I_1}{\partial p_2}+\frac{\partial I_2}{\partial x_4})(\frac{\partial H_2}{\partial x_2}-\frac{\partial H_1}{\partial p_4})=0. \quad (2.79)$$

In Hamiltons approach if one consider that the system does not depend on time t, then the term $\frac{\partial I_1}{\partial t}$ and $\frac{\partial I_2}{\partial t}$ in eq.(2.78) and eq.(2.79) are equal to zero. Simultaneously if imaginary parts I_2 and H_2 in I and H, are absent, then eqn.(2.78) and (2.79) can rewritten as

$$\frac{\partial I_1}{\partial t}+(\frac{\partial I_1}{\partial x_1})(\frac{\partial H_1}{\partial p_1})-(\frac{\partial I_2}{\partial x_1})(\frac{\partial H_2}{\partial p_1})-(\frac{\partial I_1}{\partial p_1})(\frac{\partial H_1}{\partial x_1})-(\frac{\partial I_2}{\partial p_1})(\frac{\partial H_2}{\partial x_1})$$
$$+(\frac{\partial I_1}{\partial x_2})(\frac{\partial H_1}{\partial p_2})-(\frac{\partial I_1}{\partial p_4})(\frac{\partial H_2}{\partial p_2})-(\frac{\partial I_1}{\partial p_2})(\frac{\partial H_1}{\partial x_2})-(\frac{\partial I_2}{\partial p_2})(\frac{\partial H_2}{\partial x_2})=0. \quad (2.80)$$

and

$$\frac{\partial I_2}{\partial t}+(\frac{\partial I_2}{\partial x_1})(\frac{\partial H_1}{\partial p_1})+(\frac{\partial I_1}{\partial x_1})(\frac{\partial H_2}{\partial p_1})-(\frac{\partial I_2}{\partial p_1})(\frac{\partial H_1}{\partial x_1})+(\frac{\partial I_1}{\partial p_1})(\frac{\partial H_2}{\partial x_1})$$
$$+(\frac{\partial I_2}{\partial x_2})(\frac{\partial H_1}{\partial p_2})+(\frac{\partial I_2}{\partial x_2})(\frac{\partial H_2}{\partial p_2})-(\frac{\partial I_2}{\partial p_2})(\frac{\partial H_1}{\partial x_2})+(\frac{\partial I_1}{\partial p_2})(\frac{\partial H_2}{\partial x_2})=0. \quad (2.81)$$

The recipe for the construction of complex invariant I is same as that used pervious cases for one-dimensional systems. After developing the formalism, in what follows we find invariants of for some specific real/complex potentials.

2.3.2 Harmonic oscillator system

One of the most important and widely used quantitative measures of complicated behavior of a dynamical system is the Simple harmonic oscillator system. With a motivation to construct complex invariant by using the methods discussed in previous section for some cases here, we consider simple harmonic oscillator systems. Note that for such systems

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}w^2(x^2 + y^2). \tag{2.82}$$

There exist a well known complex invariant, $u = ln(p + im\omega x) - i\omega t$ (as cited in [2]), for complex TD harmonic oscillator system. The complex version of (2.82), i.e. the $\mathcal{PT}$-symmetric Hamiltonian can attain by using (2.71) in (2.82), as $H = H_1 + iH_2$ with

$$\begin{aligned} H_1 &= \frac{1}{2}(p_1^2 - x_3^2 + p_2^2 - x_4^2) + \frac{1}{2}w^2(x_1^2 - p_3^2 - x_2^2 - p_4^2); \\ H_2 &= p_1x_3 + w^2x_1p_3 + p_2x_4 + w^2x_2p_4. \end{aligned} \tag{2.83}$$

With relate to the physical insight into the complexified version of the Hamiltonian, the following observation is worth mentioning. In fact, it is found [9] that if H_1 can be identified with the (real) Hamiltonian of a two-dimensional physical system, then in several cases (studied in Ref. [9]) the analyticity property of H suggests that H_2 is the second integral of the motion of the system in the sense that $[H_1, H_2] = 0$, and H_1 and H_2 are linearly independent. The above Hamiltonian may accept invariant I in the form

$$I = a_{01}(x, y) + a_{02}(x, y)(p_x^2 + p_y^2) + a_{11}(x, y)p_xp_y. \tag{2.84}$$

and write its complex translation in the form $I = I_1 + iI_2$ where

$$\begin{aligned} I_1 &= (a_{01xr} + a_{01yr}) + (a_{02xr} + a_{02yr})(p_1^2 - x_3^2) - (a_{02xi} + a_{02yi})(2p_1x_3) + (a_{02xr} + a_{02yr})(p_2^2 - x_4^2) \\ &-(a_{02xi} + a_{02yi})(2p_2x_4) + (a_{11xr} + a_{11yr})(p_1p_2 - x_1x_2) - (a_{11xi} + a_{11yi})(p_1x_4 - p_2x_3); \end{aligned} \tag{2.85}$$

and

$$\begin{aligned} I_2 &= (a_{01xi} + a_{01yi}) + (a_{02xi} + a_{02yi})(p_1^2 - x_3^2) + (a_{02xr} + a_{02yr})(2p_1x_3) + (a_{02yi} + a_{02xi})(p_2^2 - x_4^2) \\ &+(a_{02xr} + a_{02yr})(2p_2x_4) + (a_{11xr} + a_{11yr})(p_1x_4 - p_2x_3) + (a_{11xi} + a_{11yi})(p_1p_2 - x_4x_3). \end{aligned} \tag{2.86}$$

and the complex coefficient's $a_{01}(x, y)$, $a_{02}(x, y)$ and $a_{11}(x, y)$ are functions of (x_1, p_3, x_2, p_4) with $a_{01xr}, a_{01yr}, a_{01xi}, a_{01yi}, a_{02xr}, a_{02yr}, a_{02xi}, a_{02yi}, a_{11xr}, a_{11yr}, a_{11xi}, a_{11yi}$ are the real functions of their real arguments. Substitution of (2.83), (2.86), in (2.78) yields the expression

$$\Big[[(\frac{\partial a_{01xr}}{\partial x_1} + \frac{\partial a_{01xi}}{\partial p_3}) + (\frac{\partial a_{02xr}}{\partial x_1} + \frac{\partial a_{02xi}}{\partial p_3})(p_1^2 - x_3^2) - (2p_1x_3)(\frac{\partial a_{02xi}}{\partial x_1} - \frac{\partial a_{02xr}}{\partial p_3}) + (\frac{\partial a_{02xr}}{\partial x_1} + \frac{\partial a_{02xi}}{\partial p_3})(p_2^2 - x_4^2)$$
$$-(2p_2x_4)(\frac{\partial a_{02xi}}{\partial x_1} - \frac{\partial a_{02xr}}{\partial p_3}) - (p_1x_4 - p_2x_3)(\frac{\partial a_{11xi}}{\partial x_1} - \frac{\partial a_{11xr}}{\partial p_3}) + (p_1p_2 - x_4x_3)(\frac{\partial a_{11xr}}{\partial x_1} + \frac{\partial a_{11xi}}{\partial p_3})](2p_1)$$

$$-[(\frac{\partial a_{01xi}}{\partial x_1}-\frac{\partial a_{01xr}}{\partial p_3})+(\frac{\partial a_{02xi}}{\partial x_1}+\frac{\partial a_{02xr}}{\partial p_3})(p_1^2-x_3^2)+(2p_1x_3)(\frac{\partial a_{02xr}}{\partial x_1}+\frac{\partial a_{02xi}}{\partial p_3})+(\frac{\partial a_{02xi}}{\partial x_1}-\frac{\partial a_{02xr}}{\partial p_3})(p_2^2-x_4^2)$$
$$+(2p_2x_4)(\frac{\partial a_{02xr}}{\partial x_1}-\frac{\partial a_{02xi}}{\partial p_3})+(p_1x_4+p_2x_3)(\frac{\partial a_{11xr}}{\partial x_1}-\frac{\partial a_{11xi}}{\partial p_3})+(p_1p_2-x_4x_3)(\frac{\partial a_{11xi}}{\partial x_1}+\frac{\partial a_{11xr}}{\partial p_3})](2x_3)\Big]$$

$$-[\{4p_1(a_{02xr}+a_{02yr})-4x_3(a_{02xi}+a_{02yi})+2p_2(a_{11xr}+a_{11yr})-2x_4(a_{11xi}+a_{11yi})\}(2w^2x_1)$$
$$-\{4p_1(a_{02xi}+a_{02yi})+4x_3(a_{02xr}+a_{02yr})+2p_2(a_{11xi}+a_{11yi})+2x_4(a_{11xr}+a_{11yr})\}(2w^2p_3)]$$

$$\Big[[(\frac{\partial a_{01yr}}{\partial x_2}+\frac{\partial a_{01yi}}{\partial p_4})+(\frac{\partial a_{02yr}}{\partial x_2}+\frac{\partial a_{02yi}}{\partial p_4})(p_1^2-x_3^2)-(2p_1x_3)(\frac{\partial a_{02yi}}{\partial x_2}-\frac{\partial a_{02yr}}{\partial p_4})+(\frac{\partial a_{02yr}}{\partial x_2}+\frac{\partial a_{02yi}}{\partial p_4})(p_2^2-x_4^2)$$
$$-(2p_2x_4)(\frac{\partial a_{02yi}}{\partial x_2}-\frac{\partial a_{02yr}}{\partial p_4})-(p_1x_4-p_2x_3)(\frac{\partial a_{11yi}}{\partial x_2}-\frac{\partial a_{11yr}}{\partial p_4})+(p_1p_2-x_4x_3)(\frac{\partial a_{11yr}}{\partial x_2}+\frac{\partial a_{11yi}}{\partial p_4})](2p_2)$$

$$-[(\frac{\partial a_{01yi}}{\partial x_2}-\frac{\partial a_{01yr}}{\partial p_4})+(\frac{\partial a_{02yi}}{\partial x_2}+\frac{\partial a_{02yr}}{\partial p_4})(p_1^2-x_3^2)+(2p_1x_3)(\frac{\partial a_{02yr}}{\partial x_2}+\frac{\partial a_{02yi}}{\partial p_4})+(\frac{\partial a_{02yi}}{\partial x_2}-\frac{\partial a_{02yr}}{\partial p_4})(p_2^2-x_4^2)$$
$$+(2p_2x_4)(\frac{\partial a_{02yr}}{\partial x_2}-\frac{\partial a_{02yi}}{\partial p_4})+(p_1x_4+p_2x_3)(\frac{\partial a_{11yr}}{\partial x_2}-\frac{\partial a_{11yi}}{\partial p_4})+(p_1p_2-x_4x_3)(\frac{\partial a_{11yi}}{\partial x_2}+\frac{\partial a_{11yr}}{\partial p_4})](2x_4)\Big]$$

$$-[\{4p_2(a_{02xr}+a_{02yr})-4x_4(a_{02xi}+a_{02yi})+2p_1(a_{11xr}+a_{11yr})-2x_3(a_{11xi}+a_{11yi})\}(2w^2x_2)$$
$$-\{4p_2(a_{02xi}+a_{02yi})+4x_4(a_{02xr}+a_{02yr})+2p_1(a_{11xi}+a_{11yi})+2x_3(a_{11xr}+a_{11yr})\}(2w^2p_4)]=0.\quad(2.87)$$

which can be rationalized with respect to the power of p_1, x_3, p_2, x_4 and their fusion to give the following set of twelve coupled partial differential equations

$$(\frac{\partial a_{01xr}}{\partial x_1}+\frac{\partial a_{01xi}}{\partial p_3})-4w^2x_1(a_{02xr}+a_{02yr})+4w^2p_3(a_{02xi}+a_{02yi})=0;$$

$$(\frac{\partial a_{02xr}}{\partial x_1}+\frac{\partial a_{02xi}}{\partial p_3})=0;\qquad(\frac{\partial a_{11xr}}{\partial p_3}+\frac{\partial a_{11xi}}{\partial x_1})=0\quad(2.88)$$

$$(\frac{\partial a_{01xi}}{\partial x_1}+\frac{\partial a_{01xr}}{\partial p_3})+4w^2x_1(a_{02xi}+a_{02yi})+4w^2p_1(a_{02xr}+a_{02yr})=0;$$

$$-(\frac{\partial a_{02xi}}{\partial x_1}+\frac{\partial a_{02xr}}{\partial p_3})=0;\qquad(\frac{\partial a_{11xr}}{\partial x_1}+\frac{\partial a_{11xi}}{\partial p_3})=0\quad(2.89)$$

$$(\frac{\partial a_{01yr}}{\partial x_2}+\frac{\partial a_{01yi}}{\partial p_4})-4w^2x_2(a_{02xr}+a_{02yr})+4w^2p_4(a_{02xi}+a_{02yi})=0;$$

$$(\frac{\partial a_{02yr}}{\partial x_2}+\frac{\partial a_{02yi}}{\partial p_4})=0;\qquad(\frac{\partial a_{11yr}}{\partial x_2}-\frac{\partial a_{11yi}}{\partial p_4})=0\quad(2.90)$$

$$-(\frac{\partial a_{01yi}}{\partial x_2}-\frac{\partial a_{01yr}}{\partial p_4})+4w^2x_2(a_{02xi}+a_{02yi})+4w^2p_4(a_{02xr}+a_{02yr})=0;$$

$$-(\frac{\partial a_{02yi}}{\partial x_2}-\frac{\partial a_{02yr}}{\partial p_4})=0;\qquad(\frac{\partial a_{11yr}}{\partial p_4}+\frac{\partial a_{11yi}}{\partial x_2})=0.\quad(2.91)$$

one has to find out solutions for following unknown parameters, that renders the complex invariant.

(A) Solutions for a_{11xr}, a_{11xi}.

For determination of a_{11xr}, a_{11xi} , equations (2.88) and (2.89) can be reduced to similar second-order forms for the functions a_{11xr}, a_{11xi}, respectively,as

$$\frac{\partial^2 a_{11xr}}{\partial x_1^2} + \frac{\partial^2 a_{11xr}}{\partial p_3^2} = 0; \quad \frac{\partial^2 a_{11xi}}{\partial x_1^2} + \frac{\partial^2 a_{11xi}}{\partial p_3^2} = 0. \tag{2.92}$$

solution of (2.92) in the form

$$\begin{aligned} a_{11xr} &= \frac{\alpha}{2}(x_1^2 - p_3^2) + \alpha_1 x_1 + \alpha_2 p_3 + \delta_1; \\ a_{11xi} &= \frac{\beta}{2}(x_1^2 - p_3^2) + \beta_1 x_1 + \beta_2 p_3 + \delta_2. \end{aligned} \tag{2.93}$$

where $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \delta_1, \delta_2$ are arbitrary constant of integration to be determined later.

(B) Again for the solutions for a_{02xr}, a_{02xi} , equations (2.88) and (2.89) can be reduced to similar second-order forms for the functions a_{02xr}, a_{02xi}, respectively,as

$$\frac{\partial^2 a_{02xr}}{\partial x_1^2} + \frac{\partial^2 a_{02xr}}{\partial p_3^2} = 0; \quad \frac{\partial^2 a_{02xi}}{\partial x_1^2} + \frac{\partial^2 a_{02xi}}{\partial p_3^2} = 0. \tag{2.94}$$

it is not difficult to obtain the solution of (2.94) in the form

$$\begin{aligned} a_{02xr} &= \frac{\nu}{2}(x_1^2 - p_3^2) + \nu_1 x_1 + \nu_2 p_3 + \delta_3; \\ a_{02xi} &= \frac{\rho}{2}(x_1^2 - p_3^2) + \rho_1 x_1 + \rho_2 p_3 + \delta_4. \end{aligned} \tag{2.95}$$

where $\nu, \nu_1, \nu_2, \rho, \rho_1, \rho_2, \delta_3, \delta_4$ are arbitrary constant of integration to be determined later.

(C) Similarly to solve a_{01xr}, a_{01xi}. on differentiating (2.88) with respect to x_1 and again (2.89) with respect to p_3 and add

$$\begin{aligned} \frac{\partial^2 a_{01xr}}{\partial x_1^2} + \frac{\partial^2 a_{01xr}}{\partial p_3^2} &= (4w^2 x_1)(-2\frac{\partial a_{02xi}}{\partial p_3}) - (2\frac{\partial a_{02xi}}{\partial x_1})(4w^2 p_3) \\ &= -8w^2[x_1(-\rho p_3 + \rho_2) + p_3(\rho x_1 + \rho_1)] \\ &= -8w^2[\rho_2 x_1 + \rho_1 p_3]. \end{aligned} \tag{2.96}$$

where we have used eqn.(2.88) and then expression (2.89) to simplify the right hand side. This lead to pair of ordinary partial differential equations whose solution immediately will yield

$$a_{01xr}(x_1, p_3) = -\frac{4}{3}w^2(\rho_2 x_1^3 + \rho_1 p_3^3) + \frac{c_0}{2}(x_1^2 - p_3^2) + c_1 x_1 + c_2 p_3 + c_3. \tag{2.97}$$

where c_o, c_1, c_2 and c_3 are the arbitrary constants of integration. Another result for a_{01xr} can be obtained if one retains the term $(\frac{\partial a_{02xr}}{\partial x_1})$ and $(\frac{\partial a_{02xr}}{\partial p_3})$ in (2.96) instead of $(\frac{\partial a_{02xi}}{\partial p_3})$ and $(\frac{\partial a_{02xi}}{\partial x_1})$ and looks for the solution of the resultant partial differential equations again in the separable in the form x_1 and p_3. The corresponding result for a_{02xr} then becomes

$$\begin{aligned}
&\frac{\partial^2 a_{01xr}}{\partial x_1^2}+\frac{\partial^2 a_{01xr}}{\partial p_3^2}-(4w^2x_1)(\frac{\partial a_{02xr}}{\partial x_1}-\frac{\partial a_{02xi}}{\partial p_3})+(4w^2p_3)(\frac{\partial a_{02xi}}{\partial x_1}+\frac{\partial a_{02xr}}{\partial p_3})=0.\\
=&\ 4w^2x_1(2\frac{\partial a_{02xr}}{\partial x_1})-4w^2p_3(\frac{\partial a_{02xr}}{\partial p_3})\\
=&\ 8w^2[x_1(\nu x_1+\nu_1)-p_3(-\nu p_3+\nu_2)]\\
=&\ 8w^2[\nu(x_1^2+p_3^2)+\nu_1x_1-\nu_2p_3]. \qquad (2.98)
\end{aligned}$$

From above equation we have

$$a_{01xr}(x_1,p_3)=\frac{2}{3}w^2\nu(x_1^4+p_3^4)+\frac{4}{3}w^2(\nu_1x_1^3-\nu_2p_3^3)+\frac{\bar{c_0}}{2}(x_1^2-p_3^2)+\bar{c_1}x_1+\bar{c_2}p_3+\bar{c_3}. \qquad (2.99)$$

Uniqueness of expression (2.97) and (2.99) for a_{01xr} would require that

$$\nu=0,\nu_2=\rho_1,\nu_1=-\rho_2,c_0=\bar{c_0},c_1=\bar{c_1},c_2=\bar{c_2},c_3=\bar{c_3}. \qquad (2.100)$$

This will yield acceptable form of a_{01xr} as

$$a_{01xr}(x_1,p_3)=-\frac{4}{3}w^2(\rho_2x_1^3+\rho_1p_3^3)+c_0(x_1^2-p_3^2)+c_1x_1+c_2p_3+c_3. \qquad (2.101)$$

For determination of a_{01xi} one follows the same procedure as followed for a_{01xr} and obtains the coefficient function $a_{01xr}(x_1,p_3)$ in the form

$$a_{01xi}(x_1,p_3)=\frac{4}{3}w^2(\rho_1x_1^3-\rho_2p_3^3)+d_0(x_1^2-p_3^2)+d_1x_1+d_2p_3+d_3. \qquad (2.102)$$

where d_o,d_1,d_2 and d_3 are the arbitrary constants of integration. With the choice (2.100) (including now $\nu=0$)for the arbitrary constants the expressions a_{02xr} and a_{02xi} now takes the form

$$a_{02xr}=-\rho_2x_1+\rho_1p_3+\delta_3;\qquad a_{02xi}=\rho_1x_1+\rho_2p_3+\delta_4.$$

In the similar methods we can have the expressions a_{11xr} and a_{11xi} will now takes the form

$$a_{11xr}=-\beta_2x_1+\beta_1p_3+\delta_1;\qquad a_{11xi}=+\beta_1x_1+\beta_2p_3+\delta_2 \qquad (2.103)$$

(D) Solutions for a_{11yr},a_{11yi}

For determination of a_{11yr},a_{11yi}, equations (2.90) and (2.91) can be reduced to similar second-order forms for the functions a_{11yr},a_{11yi}, respectively as

$$\frac{\partial^2 a_{11yr}}{\partial x_2^2}+\frac{\partial^2 a_{11yr}}{\partial p_4^2}=0;\qquad \frac{\partial^2 a_{11yi}}{\partial x_2^2}+\frac{\partial^2 a_{11yi}}{\partial p_4^2}=0. \qquad (2.104)$$

solution of (2.104) in the form

$$\begin{aligned}
a_{11yr}&=\frac{\alpha}{2}(x_2^2-p_4^2)+\alpha_1x_2+\alpha_2p_4+\delta_5;\\
a_{11yi}&=\frac{\beta}{2}(x_2^2-p_4^2)+\beta_1x_2+\beta_2p_4+\delta_6. \qquad (2.105)
\end{aligned}$$

where $\alpha, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \delta_5, \delta_6$ are arbitrary constant of integration to be determined later.

(E) solutions for a_{02yr}, a_{02yi}

Equations (2.90) and (2.91) can be reduced to similar second-order forms for the functions a_{02yr}, a_{02yi}, respectively as

$$\frac{\partial^2 a_{02yr}}{\partial x_2^2} + \frac{\partial^2 a_{02yr}}{\partial p_4^2} = 0; \qquad \frac{\partial^2 a_{02yi}}{\partial x_2^2} + \frac{\partial^2 a_{02yi}}{\partial p_4^2} = 0. \tag{2.106}$$

solution of (2.106) in the form

$$\begin{aligned} a_{02yr} &= \frac{\nu}{2}(x_1^2 - p_3^2) + \nu_1 x_1 + \nu_2 p_3 + \delta_7; \\ a_{02yi} &= \frac{\rho}{2}(x_1^2 - p_3^2) + \rho_1 x_1 + \rho_2 p_3 + \delta_8. \end{aligned} \tag{2.107}$$

where $\nu, \rho, \nu_1, \nu_2, \rho_1, \rho_2, \delta_3, \delta_4$ are arbitrary constant of integration to be determined later.

(F) Similarly to solve a_{01yr}, a_{01yi}, on differentiating (2.90) with respect to x_2 and (2.91) with respect to p_4 and add

$$\begin{aligned} \frac{\partial^2 a_{01yr}}{\partial x_2^2} + \frac{\partial^2 a_{01yr}}{\partial p_4^2} &= (4w^2 x_2)(-2\frac{\partial a_{02yi}}{\partial p_4}) - (2\frac{\partial a_{02yi}}{\partial x_2})(4w^2 p_4) \\ &= -8w^2[x_2(-\rho p_3 + \rho_2) + p_4(\rho x_2 + \rho_1)] \\ &= -8w^2[\rho_2 x_2 + \rho_1 p_4]. \end{aligned} \tag{2.108}$$

where we have used eqn.(2.90) and then expression (2.107) to simplify the right hand side. This lead to pair of ordinary partial differential equations whose solution immediately will yield

$$a_{01yr}(x_2, p_4) = \frac{4}{3}w^2(\rho_2 x_2^3 + \rho_1 p_4^3) + \frac{c_0}{2}(x_2^2 - p_4^2) + c_1 x_2 + c_2 p_4 + c_3. \tag{2.109}$$

where c_o, c_1, c_2 and c_3 are the arbitrary constants of integration. Another result for a_{01yr} can be obtained if one retains the term $(\frac{\partial a_{02yr}}{\partial x_2})$ and $(\frac{\partial a_{02yr}}{\partial p_4})$ in (2.108) instead of $(\frac{\partial a_{02yi}}{\partial p_4})$ and $(\frac{\partial a_{02yi}}{\partial x_2})$ and looks for the solution of the resultant partial differential equations again in the separable in the form x_2 and p_4.The corresponding result for a_{02yr} then becomes

$$\begin{aligned} & \frac{\partial^2 a_{01yr}}{\partial x_2^2} + \frac{\partial^2 a_{01yr}}{\partial p_4^2} - (4w^2 x_2)(\frac{\partial a_{02yr}}{\partial x_2} - \frac{\partial a_{02yi}}{\partial p_4}) + (4w^2 p_4)(\frac{\partial a_{02yi}}{\partial x_2} + \frac{\partial a_{02yr}}{\partial p_4}) = 0 \\ = \; & (4w^2 x_2)(2\frac{\partial a_{02yr}}{\partial x_2}) - (4w^2 p_4)(\frac{\partial a_{02yr}}{\partial p_4}) \\ = \; & 8w^2[x_2(\nu x_2 + \nu_1) - p_4(-\nu p_4 + \nu_2)] \\ = \; & 8w^2[\nu(x_2^2 + p_4^2) + \nu_1 x_2 - \nu_2 p_4]. \end{aligned} \tag{2.110}$$

From above equation we have

$$a_{01yr}(x_2, p_4) = \frac{2}{3}w^2\nu(x_2^4 + p_4^4) + \frac{4}{3}w^2(\nu_1 x_2^3 - \nu_3 p_4^3) + \frac{\bar{c_0}}{2}(x_2^2 - p_4^2) + \bar{c_1} x_2 + \bar{c_2} p_4 + \bar{c_3}. \tag{2.111}$$

Uniqueness of expression (2.109) and (2.111) for a_{01yr} would require that

$$\nu = 0, \nu_2 = \rho_1, \nu_1 = -\rho_2, c_0 = \bar{c_0}, c_1 = \bar{c_1}, c_2 = \bar{c_2}, c_3 = \bar{c_3}. \tag{2.112}$$

This will yield acceptable form of a_{01yr} as

$$a_{01yr}(x_2, p_4) = -\frac{4}{3}w^2(\rho_2 x_2^3 + \rho_1 p_4^3) + c_0(x_2^2 - p_4^2) + c_1 x_2 + c_2 p_4 + c_3. \tag{2.113}$$

For determination of a_{01yi} one follows the same procedure as followed for a_{01xr} and obtains the coefficient function $a_{01yr}(x_2, p_4)$ in the form

$$a_{01yi}(x_2, p_4) = \frac{4}{3}w^2(\rho_1 x_2^3 - \rho_2 p_4^3) + d_0(x_2^2 - p_4^2) + d_1 x_2 + d_2 p_4 + d_3. \tag{2.114}$$

where d_o, d_1, d_2 and d_3 are the arbitrary constants of integration. With the choice (2.112) (including now $\nu = 0$)for the arbitrary constants the expressions a_{02yr} and a_{02yi}

$$a_{02yr} = -\rho_2 x_2 + \rho_1 p_4 + \delta_7; \quad a_{02yi} = +\rho_1 x_2 + \rho_2 p_4 + \delta_8.$$

Again in similar method we can find the expressions a_{11yr} and a_{11yi} now takes the form

$$a_{11yr} = -\beta_2 x_2 + \beta_1 p_4 + \delta_5; \quad a_{11yi} = +\beta_1 x_2 + \beta_2 p_4 + \delta_6. \tag{2.115}$$

Note that the forms eqs. [(2.88)-(2.91)] of $a_{01xr}, a_{01yr}, a_{01xi}, a_{01yi}, a_{02xr}, a_{02yr}, a_{02xi}$, $a_{02yi}, a_{11xr}, a_{11yr}, a_{11xi}$ and a_{11yi} are determined only from (2.78). With this expressions for the coefficient function when (2.79) is rationalized, one obtained several constrains relations among the arbitrary constants appearing in eqn. [(2.88)-(2.91)], thereby reducing the number of arbitrary constants in the final solutions. The constrained so obtained are $c_1 = -d_2,\ d_1 = c_2,\ c_0 = 4w^2\delta_3 = 4w^2\delta_7,\ d_0 = 4w^2\delta_4 = 4w^2\delta_8$. which gives rise to forms of coefficient functions as

$$\begin{aligned}
&a_{11xr} = -\beta_2 x_1 + \beta_1 p_3 + \delta_1; \quad a_{11xi} = \beta_1 x_1 + \beta_2 p_3 + \delta_2;\\
&a_{02xr} = -\rho_2 x_1 + \rho_1 p_3 + \delta_3; \quad a_{02xi} = \rho_1 x_1 + \rho_2 p_3 + \delta_4;\\
&a_{01xr}(x_1, p_3) = -\frac{4}{3}w^2(\rho_2 x_1^3 + \rho_1 p_3^3) + c_0(x_1^2 - p_3^2) + c_1 x_1 + d_1 p_3 + c_3;\\
&a_{01xi}(x_1, p_3) = \frac{4}{3}w^2(\rho_1 x_1^3 - \rho_2 p_3^3) + d_0(x_1^2 - p_3^2) + d_1 x_1 - c_1 p_3 + d_3;\\
&a_{11yr} = -\beta_2 x_2 + \beta_1 p_4 + \delta_5; \quad a_{11yi} = +\beta_1 x_2 + \beta_2 p_4 + \delta_6;\\
&a_{02yr} = -\rho_2 x_2 + \rho_1 p_4 + \delta_7; \quad a_{02yi} = +\rho_1 x_2 + \rho_2 p_4 + \delta_8;\\
&a_{01yr}(x_2, p_4) = -\frac{4}{3}w^2(\rho_2 x_2^3 + \rho_1 p_4^3) + c_0(x_2^2 - p_4^2) + c_1 x_2 + d_1 p_4 + c_3;\\
&a_{01yi}(x_2, p_4) = \frac{4}{3}w^2(\rho_1 x_2^3 - \rho_2 p_4^3) + c_0(x_2^2 - p_4^2) + d_1 x_2 - c_1 p_4 + c_3.
\end{aligned} \tag{2.116}$$

So from above set of equations, we have

$$\begin{aligned}
&a_{11r} = a_{11xr} + a_{11yr} = -\beta_2(x_1 + x_2) + \beta_1(p_3 + p_4);\\
&a_{11i} = a_{11xi} + a_{11yi} = \beta_1(x_1 + x_2) + \beta_2(p_3 + p_4);\\
&a_{02r} = a_{02xr} + a_{02yr} = -\rho_2(x_1 + x_2) + \rho_1(p_3 + p_4) + \frac{c_0}{2w^2};
\end{aligned}$$

$$a_{02i} = a_{02xi} + a_{02yi} = \rho_1(x_1 + x_2) + \rho_2(p_3 + p_4) + \frac{d_0}{2w^2};$$
$$a_{01r} = -\frac{4}{3}w^2[(\rho_2 x_1^3 + \rho_2 x_2^3) + (\rho_1 p_3^3 + \rho_1 p_4^3)] + \frac{c_0}{2}(x_1^2 + x_2^2 - p_3^2 - p_4^2) + c_1(x_1 + x_2) + d_1(p_3 + p_4);$$
$$a_{01i} = \frac{4}{3}w^2[\rho_1(x_1^3 + x_2^3) - \rho_2(p_3^3 + p_4^3)] + \frac{d_0}{2}(x_1^2 + x_2^2 - p_3^2 - p_4^2) + d_1(x_1 + x_2) - c_1(p_3 + p_4). \quad (2.117)$$

Construction of Invariants. For the construction of complex invariants using the results (2.117) for $a_{01r}, a_{01i}, a_{02r}, a_{02i}, a_{11r}, a_{11i}$ one can obtain the complex invariant I from (2.85)and (2.86), in which the real and imaginary parts I_1 and I_2 are given by

$$I_1 = \frac{4}{3}w^2[\rho_2(x_1^3 + x_2^3) + \rho_1(p_3^3 + p_4^3)] + \frac{c_0}{2}(x_1^2 + x_2^2 - p_3^2 - p_4^2) + c_1(x_1 + x_2) + d_1(p_3 + p_4)$$
$$-2(p_1x_3 + p_4x_4)\{\beta_1(x_1 + x_2) + \beta_2(p_3 + p_4)\} + (p_1^2 + p_2^2 - x_3^2 - x_4^2)\{-\beta_2(x_1 + x_2) + \beta_1(p_3 + p_4)\}$$
$$+(p_1p_2 - x_3x_4)\{-\rho_2(x_1 + x_2) + \rho_1(p_3 + p_4)\} - (p_1x_4 + p_2x_3)\{\rho_1(x_1 + x_2) + \rho_2(p_3 + p_4)\},$$

$$I_2 = \frac{4}{3}w^2[\rho_1(x_1^3 + x_2^3) - \rho_2(p_3^3 + p_4^3)] + \frac{d_0}{2}(x_1^2 + x_2^2 - p_3^2 - p_4^2) + d_1(x_1 + x_2) - c_1(p_3 + p_4)$$
$$+2(p_1x_3 + p_4x_4)\{-\beta_2(x_1 + x_2) + \beta_1(p_3 + p_4)\} + (p_1^2 + p_2^2 - x_3^2 - x_4^2)\{\beta_1(x_1 + x_2) + \beta_2(p_3 + p_4)\}$$
$$+(p_1p_2 - x_3x_4)\{\rho_1(x_1 + x_2) + \rho_2(p_3 + p_4)\} + (p_1x_4 + p_2x_3)\{-\rho_2(x_1 + x_2) + \rho_1(p_3 + p_4)\}. \quad (2.118)$$

Finally the complex invariant $I = I_1 + iI_2$ can be written as

$$I = \frac{4}{3}w^2b[(x_1^3 + x_2^3) + i(p_3^3 + p_4^3)] + \frac{\sigma_0}{2}(x_1^2 + x_2^2 - p_3^2 - p_4^2) + \sigma_1(x_1 + x_2) - i(p_3 + p_4)$$
$$+(p_1^2 - x_3^2 + 2ip_1x_3 + p_2^2 - x_4^2 + 2ip_4x_4)b[(x_1 + x_2) - i(p_3 + p_4) + \frac{\sigma_0}{2w^2}]$$
$$+[(x_1 + x_2) - i(p_3 + p_4)]e(p_1p_2 - x_3x_4) + (p_1x_4 + p_2x_3)e[i(x_1 + x_2) + (p_3 + p_4)].$$

or on simplification we can write

$$I = \frac{4}{3}w^2b[(x_1^3 + x_2^3) + i(p_3^3 + p_4^3)] + \frac{\sigma_0}{2}(x_1^2 + x_2^2 - p_3^2 - p_4^2) + \sigma_1(x_1 + x_2) - i(p_3 + p_4)$$
$$+(p_1^2 - x_3^2 + 2ip_1x_3 + p_2^2 - x_4^2 + 2ip_4x_4)b[(x_1 + x_2) - i(p_3 + p_4) + \frac{\sigma_0}{2w^2}]$$
$$+[(x_1 + x_2) - i(p_3 + p_4)]e(p_1p_2 - x_3x_4 + ip_1x_4 + ip_2x_3). \quad (2.119)$$

where $e = \rho_2 + i\rho_1$; $\sigma_0 = c_0 + id_0$; $\sigma_1 = c_1 + id_1$ and $= -\beta_2 + i\beta_1$ are the arbitrary constants. In real and complex coordinate it is written as

$$I = \frac{w^2b}{3}[x^*(3x^2 + x^*) + y^*(3y^2 + y^*)] + \frac{\sigma_0}{4}(x^2 + y^2 + x^{*2} + y^{*2}) + \sigma_1(x^* + y^*)$$
$$+b[(x^* + y^*) + \frac{\sigma_0}{2w^2}](p_x^{*2} + p_y^{*2}) + e(x^* + y^*)p_xp_y.$$

where $x^* = x_1 - ip_3$; $y^* = x_2 - ip_4$; $p_x^* = p_1 - ix_3$; $p_y^* = p_2 - ix_4$. are as defined earlier, which conforms the invariance condition for integrability with view of poisson bracket.

2.3.3 Non-linear oscillator system with cubic potential

With a view to constructing complex invariant for some cases here, in this sections we make the use of methods discussed in previous section. Spectral analysis of the complex cubic oscillator has bee carried out by many researchers [8]. Using the exact semiclassical analysis, They studied the spectrum of a one-parameter family of complex cubic oscillators. Complex dynamical invariants are searched out for two dimensional complex potentials using rationalization method within the framework of an extended complex phase space. Consider the case of non-hermitian $\mathcal{PT}$-symmetric Hamiltonian systems with in the frame work of rationalization method. Note that for duffing oscillator systems

$$H = \frac{1}{2}(p_x^2 + p_y^2) + i(x + y) + i(x^3 + y^3). \tag{2.120}$$

We can rearrange to get the the complex version of (2.120), by using (2.71) in (2.120), as

$$\begin{aligned} H_1 &= \frac{1}{2}p_1^2 - \frac{1}{2}x_3^2 + \frac{1}{2}p_2^2 - \frac{1}{2}x_4^2 - p_3 - p_4 + p_3^3 + p_4^3 - 3x_1^2p_3 - 3x_2^2p_4; \\ H_2 &= p_1x_3 + p_2x_4 + x_1 + x_2 + x_1^3 + x_2^3 - 3x_1p_3^2 - 3x_2p_4^2. \end{aligned} \tag{2.121}$$

The above systems comply the complex invariant I in the form

$$I = a_{01}(x, y) + a_{02}(x, y)(p_x^2 + p_y^2) + a_{11}(x, y)p_xp_y. \tag{2.122}$$

and write its complex version in the form $I = I_1 + iI_2$ where

$$\begin{aligned} I_1 &= (a_{01xr} + a_{01yr}) + (a_{02xr} + a_{02yr})(p_1^2 - x_3^2) - (a_{02xi} + a_{02yi})(2p_1x_3) + (a_{02xr} + a_{02yr})(p_2^2 - x_4^2) \\ &-(a_{02xi} + a_{02yi})(2p_2x_4) + (a_{11xr} + a_{11yr})(p_1p_2 - x_1x_2) - (a_{11xi} + a_{11yi})(p_1x_4 - p_2x_3); \end{aligned} \tag{2.123}$$

and

$$\begin{aligned} I_2 &= (a_{01xi} + a_{01yi}) + (a_{02xi} + a_{02yi})(p_1^2 - x_3^2) + (a_{02xr} + a_{02yr})(2p_1x_3) + (a_{02yi} + a_{02xi})(p_2^2 - x_4^2) \\ &+(a_{02xr} + a_{02yr})(2p_2x_4) + (a_{11xr} + a_{11yr})(p_1x_4 - p_2x_3) + (a_{11xi} + a_{11yi})(p_1p_2 - x_4x_3). \end{aligned} \tag{2.124}$$

Substitution of (2.121), (2.123), (2.124) in (2.78) yields the expression and rationalization of the resultant expressions with respect to the powers of p_1, x_3, p_2, x_4 and their combinations gives a set of following twelve coupled partial differential equations

$$\begin{aligned} &(\frac{\partial a_{01xr}}{\partial x_1} + \frac{\partial a_{01xi}}{\partial p_3}) - 2(1 + 3x_1^2 - 3p_3^2)(a_{02xr} + a_{02yr}) + 12x_1p_3(a_{02xi} + a_{02yi}) = 0; \\ &(\frac{\partial a_{02xr}}{\partial x_1} + \frac{\partial a_{02xi}}{\partial p_3}) = 0; \quad (\frac{\partial a_{11xr}}{\partial p_3} + \frac{\partial a_{11xi}}{\partial x_1}) = 0; \qquad (2.125) \\ &(\frac{\partial a_{01xi}}{\partial x_1} + \frac{\partial a_{01xr}}{\partial p_3}) + 2(1 + 3x_1^2 - 3p_3^2)(a_{02xi} + a_{02yi}) + 12x_1p_3(a_{02xr} + a_{02yr}) = 0; \end{aligned}$$

$$-(\frac{\partial a_{02xi}}{\partial x_1}+\frac{\partial a_{02xr}}{\partial p_3})=0; \quad (\frac{\partial a_{11xr}}{\partial x_1}+\frac{\partial a_{11xi}}{\partial p_3})=0; \tag{2.126}$$

$$(\frac{\partial a_{01yr}}{\partial x_2}+\frac{\partial a_{01yi}}{\partial p_4})-2(1+3x_2^2-3p_4^2)(a_{02xr}+a_{02yr})+12x_2p_4(a_{02xi}+a_{02yi})=0;$$

$$(\frac{\partial a_{02yr}}{\partial x_2}+\frac{\partial a_{02yi}}{\partial p_4})=0; \quad (\frac{\partial a_{11yr}}{\partial x_2}-\frac{\partial a_{11yi}}{\partial p_4})=0; \tag{2.127}$$

$$-(\frac{\partial a_{01yi}}{\partial x_2}-\frac{\partial a_{01yr}}{\partial p_4})+2(1+3x_2^2-3p_4^2)(a_{02xi}+a_{02yi})+12x_2p_4(a_{02xr}+a_{02yr})=0;$$

$$-(\frac{\partial a_{02yi}}{\partial x_2}-\frac{\partial a_{02yr}}{\partial p_4})=0; \quad (\frac{\partial a_{11yr}}{\partial p_4}+\frac{\partial a_{11yi}}{\partial x_2})=0. \tag{2.128}$$

So for construction of complex invariants one has to find out solutions for these unknown parameters

(A) Solutions for a_{11xr}, a_{11xi}. solution of a_{11xr}, a_{11xi} are in the form

$$a_{11xr}=\frac{\alpha}{2}(x_1^2-p_3^2)+\alpha_1x_1+\alpha_2p_3+\delta_1; \quad a_{11xi}=\frac{\beta}{2}(x_1^2-p_3^2)+\beta_1x_1+\beta_2p_3+\delta_2. \tag{2.129}$$

where $\alpha, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2$ are arbitrary constant of integration to be determined later.

(B) solutions for a_{02xr}, a_{02xi}. solution of a_{02xr}, a_{02xi} are in the form

$$a_{02xr}=\frac{\nu}{2}(x_1^2-p_3^2)+\nu_1x_1+\nu_2p_3+\delta_3; \quad a_{02xi}=\frac{\rho}{2}(x_1^2-p_3^2)+\rho_1x_1+\rho_2p_3+\delta_4. \tag{2.130}$$

where $\nu, \rho, \nu_1, \nu_2, \rho_1, \rho_2, \delta_3, \delta_4$ are arbitrary constant of integration to be determined later.

(C) solutions for a_{01xr}, a_{01xi}. Similarly to solve a_{01xr}, a_{01xi}, on differentiating (2.125) with respect to x_1 and again (2.125) with respect to p_3 and add

$$\begin{aligned}\frac{\partial^2 a_{01xr}}{\partial x_1^2}+\frac{\partial^2 a_{01xr}}{\partial p_3^2} &= 2(1+3x_1^2-3p_3^2)(\frac{\partial a_{02xr}}{\partial x_1}-\frac{\partial a_{02xi}}{\partial p_3})-(12x_1p_3)(\frac{\partial a_{02xi}}{\partial x_1}+\frac{\partial a_{02xr}}{\partial p_3}) \\ &= 24\rho_2x_1p_3-4\rho_1(3x_1^2-3p_3^2+1).\end{aligned}$$

where we have used eqn.(2.125) and (2.126) and then expression (2.130) (using constraints) to simplify the right hand side. This lead to pair of ordinary partial differential equations whose solution immediately will yield

$$a_{01xr}(x_1,p_3)=2\rho_2(x_1^3p_3+x_1p_3^3)-\rho_1(x_1^4-p_3^4)-\rho_1(x_1^2+p_3^2). \tag{2.131}$$

For determination of a_{01xi} one follows the same procedure as followed for a_{01xr} and obtains the coefficient function $a_{01xr}(x_1,p_3)$ in the form

$$a_{01xi}(x_1,p_3)=-2\rho_1(x_1^3p_3+x_1p_3^3)-\rho_2(x_1^4-p_3^4)-\rho_2(x_1^2+p_3^2) \tag{2.132}$$

(D) Solutions for a_{11yr}, a_{11yi}. solution of a_{11yr}, a_{11yi} are in the form

$$a_{11yr}=\frac{\alpha}{2}(x_2^2-p_4^2)+\alpha_1x_2+\alpha_2p_4+\delta_5, \quad a_{11yi}=\frac{\beta}{2}(x_2^2-p_4^2)+\beta_1x_2+\beta_2p_4+\delta_6. \tag{2.133}$$

(E) Solutions for a_{02yr}, a_{02yi}. solution of a_{02yr}, a_{02yi} are in the form

$$a_{02yr} = \frac{\nu}{2}(x_1^2 - p_3^2) + \nu_1 x_1 + \nu_2 p_3 + \delta_7, \quad a_{02yi} = \frac{\rho}{2}(x_1^2 - p_3^2) + \rho_1 x_1 + \rho_2 p_3 + \delta_8. \tag{2.134}$$

(F) Similarly to solve a_{01yr}, a_{01yi}. on differentiating (2.127) with respect to x_2 and (2.128) with respect to p_4 and add

$$\begin{aligned}\frac{\partial^2 a_{01yr}}{\partial x_2^2} + \frac{\partial^2 a_{01yr}}{\partial p_4^2} &= 2(1 + 3x_2^2 - 3p_4^2)(\frac{\partial a_{02yr}}{\partial x_2} - \frac{\partial a_{02yi}}{\partial p_4}) - (\frac{\partial a_{02yi}}{\partial x_2} + \frac{\partial a_{02yr}}{\partial p_4})(12x_2 p_4) \\ &= 24\rho_2 x_1 p_3 - 4\rho_1(3x_1^2 - 3p_3^2 + 1).\end{aligned} \tag{2.135}$$

where we have used eqn.(2.127) and (2.128) and then expression (2.134) (using constraints) to simplify the right hand side. This lead to pair of ordinary partial differential equations whose solution immediately will yield

$$a_{01yr}(x_2, p_4) = 2\rho_2(x_2^3 p_4 + x_2 p_4^3) - \rho_1(x_2^4 - p_4^4) - \rho_1(x_2^2 + p_4^2). \tag{2.136}$$

For determination of a_{01yi} one follows the same procedure as followed for a_{01xr} and obtains the coefficient function $a_{01yr}(x_2, p_4)$ in the form

$$a_{01yi}(x_2, p_4) = -2\rho_1(x_1^3 p_3 + x_1 p_3^3) - \rho_2(x_1^4 - p_3^4) - \rho_2(x_1^2 + p_3^2). \tag{2.137}$$

Here from eqs.[(2.129)-(2.137)] for co-efficient are determined only from (2.78). With this expressions for the coefficient function when (2.79) is rationalized, one obtained several constrains relations, thereby reducing the number of arbitrary constants in the final solutions. The constrained so obtained are $\delta_5 = \delta_6 = \delta_7 = \delta_8 = 0$ $\rho_1 = -\nu_2$, $\rho_2 = \nu_1$, $\beta_2 = -\alpha_1$, $\beta_1 = \alpha_2$. which gives rise to forms of coefficient functions as

$$\begin{aligned}&a_{11r} = -\beta_2(x_1 + x_2) + \beta_1(p_3 + p_4); \quad a_{11i} = \beta_1(x_1 + x_2) + \beta_2(p_3 + p_4); \\ &a_{02r} = -\rho_2(x_1 + x_2) + \rho_1(p_3 + p_4); \quad a_{02i} = \rho_1(x_1 + x_2) + \rho_2(p_3 + p_4); \\ &a_{01r} = 2\rho_2(x_1^3 p_3 + x_1 p_3^3 + x_2^3 p_4 + x_2 p_4^3) - \rho_1(x_1^4 + x_2^4 - p_3^4 - p_4^4) - \rho_1(x_1^2 + x_2^2 + p_3^2 + p_4^2); \\ &a_{01i} = -2\rho_1(x_1^3 p_3 + x_1 p_3^3 + x_2^3 p_4 + x_2 p_4^3) - \rho_2(x_1^4 + x_2^4 - p_3^4 - p_4^4) - \rho_2(x_1^2 + x_2^2 + p_3^2 + p_4^2)\end{aligned} \tag{2.138}$$

Construction of Invariants. using the co-efficients the I_1 and I_2 are given by

$$\begin{aligned}&I_1 = 4\rho_2(x_1^3 p_3 + x_1 p_3^3 + x_2^3 p_4 + x_2 p_4^3) - \rho_1(x_1^4 + x_2^4 - p_3^4 - p_4^4) - \rho_1(x_1^2 + x_2^2 + p_3^2 + p_4^2) \\ &-2(p_1 x_3 + p_4 x_4)\{\beta_1(x_1 + x_2) + \beta_2(p_3 + p_4)\} + (p_1^2 + p_2^2 - x_3^2 - x_4^2)\{-\beta_2(x_1 + x_2) + \beta_1(p_3 + p_4)\} \\ &+(p_1 p_2 - x_3 x_4)\{-\rho_2(x_1 + x_2) + \rho_1(p_3 + p_4)\} - (p_1 x_4 + p_2 x_3)\{\rho_1(x_1 + x_2) + \rho_2(p_3 + p_4)\},\end{aligned} \tag{2.139}$$

$$\begin{aligned}&I_2 = -4\rho_1(x_1^3 p_3 + x_1 p_3^3 + x_2^3 p_4 + x_2 p_4^3) - \rho_2(x_1^4 + x_2^4 - p_3^4 - p_4^4) - \rho_2(x_1^2 + x_2^2 + p_3^2 + p_4^2) \\ &+2(p_1 x_3 + p_4 x_4)\{-\beta_2(x_1 + x_2) + \beta_1(p_3 + p_4)\} + (p_1^2 + p_2^2 - x_3^2 - x_4^2)\{\beta_1(x_1 + x_2) + \beta_2(p_3 + p_4)\} \\ &+(p_1 p_2 - x_3 x_4)\{\rho_1(x_1 + x_2) + \rho_2(p_3 + p_4)\} + (p_1 x_4 + p_2 x_3)\{-\rho_2(x_1 + x_2) + \rho_1(p_3 + p_4)\}.\end{aligned} \tag{2.140}$$

Finally the complex invariant $I = I_1 + iI_2$ can be written as

$$I = ib(x^3x^* + y^3y^*) + ib(xx^* + yy^*) + \frac{b}{2}(x^* + y^*)(p_x^{*2} + p_y^{*2}) + b(x^* + y^*)p_xp_y. \tag{2.141}$$

2.3.4 Non-hermitian $\mathcal{PT}$-symmetric Hamiltonian system

Consider the case of non-$\mathcal{PT}$-symmetric Hamiltonian (C.M. Bender type [7]) systems.

$$H = (p_x^2 + p_y^2) + (x + y) + i(x^3 + y^3). \tag{2.142}$$

One can easily derive the complex version of (2.142), by using (2.71) in (2.142), as $H = H_1 + iH_2$ with

$$\begin{aligned} H_1 &= \frac{1}{2}p_1^2 - \frac{1}{2}x_3^2 + \frac{1}{2}p_2^2 - \frac{1}{2}x_4^2 + x_1 + x_2 + p_3^3 + p_4^3 - 3x_1^2p_3 - 3x_2^2p_4; \\ H_2 &= p_1x_3 + p_2x_4 + p_3 + p_4 + x_1^3 + x_2^3 - 3x_1p_3^2 - 3x_2p_4^2. \end{aligned} \tag{2.143}$$

Systems endorse a complex invariant I in the form

$$I = a_{01}(x, y) + a_{02}(x, y)(p_x^2 + p_y^2) + a_{11}(x, y)p_xp_y. \tag{2.144}$$

and write its complex version in the form $I = I_1 + iI_2$. Substitution of (2.143), (and I_1 and I_2 as discussed in previous example) in (2.78), and rationalizing with respect to the power of p_1, x_3, p_2, x_4 and their combinations the resultant expression to give the following set of twelve coupled partial differential equations;

$$\left(\frac{\partial a_{01xr}}{\partial x_1} + \frac{\partial a_{01xi}}{\partial p_3}\right) - 4(1 - 6x_1p_3)(a_{02xr} + a_{02yr}) + 4(3x_1^2 - 3p_3^2)(a_{02xi} + a_{02yi}) = 0;$$

$$\left(\frac{\partial a_{02xr}}{\partial x_1} + \frac{\partial a_{02xi}}{\partial p_3}\right) = 0; \quad a\left(\frac{\partial a_{11xr}}{\partial p_3} + \frac{\partial a_{11xi}}{\partial x_1}\right) = 0; \tag{2.145}$$

$$\left(\frac{\partial a_{01xi}}{\partial x_1} + \frac{\partial a_{01xr}}{\partial p_3}\right) + 4(1 - 6x_1p_3)(a_{02xi} + a_{02yi}) + 4(3x_1^2 - 3p_3^2)(a_{02xr} + a_{02yr}) = 0$$

$$-\left(\frac{\partial a_{02xi}}{\partial x_1} + \frac{\partial a_{02xr}}{\partial p_3}\right) = 0; \quad \left(\frac{\partial a_{11xr}}{\partial x_1} + \frac{\partial a_{11xi}}{\partial p_3}\right) = 0; \tag{2.146}$$

$$\left(\frac{\partial a_{01yr}}{\partial x_2} + \frac{\partial a_{01yi}}{\partial p_4}\right) - 4(1 - 6x_2p_4)(a_{02xr} + a_{02yr}) + 4(3x_2^2 - 3p_4^2)(a_{02xi} + a_{02yi}) = 0$$

$$\left(\frac{\partial a_{02yr}}{\partial x_2} + \frac{\partial a_{02yi}}{\partial p_4}\right) = 0; \quad \left(\frac{\partial a_{11yr}}{\partial x_2} - \frac{\partial a_{11yi}}{\partial p_4}\right) = 0; \tag{2.147}$$

$$-\left(\frac{\partial a_{01yi}}{\partial x_2} - \frac{\partial a_{01yr}}{\partial p_4}\right) + 4(1 - 6x_2p_4)(a_{02xi} + a_{02yi}) + 4(3x_2^2 - 3p_4^2)(a_{02xr} + a_{02yr}) = 0;$$

$$-\left(\frac{\partial a_{02yi}}{\partial x_2} - \frac{\partial a_{02yr}}{\partial p_4}\right) = 0; \quad \left(\frac{\partial a_{11yr}}{\partial p_4} + \frac{\partial a_{11yi}}{\partial x_2}\right) = 0. \tag{2.148}$$

These PDE gives rise to forms of coefficient functions as

$$a_{01r} = 4\rho_2(x_1^3p_3 + x_1p_3^3 + x_2^3p_4 + x_2p_4^3) + 2\rho_1(x_1^4 + x_2^4 - p_3^4 - p_4^4) - 2\rho_2(x_1^2 + x_2^2 + p_3^2 + p_4^2);$$
$$a_{01i} = -4\rho_1(x_1^3p_3 + x_1p_3^3 + x_2^3p_4 + x_2p_4^3) - 2\rho_2(x_1^4 + x_2^4 - p_3^4 - p_4^4) + 2\rho_1(x_1^2 + x_2^2 + p_3^2 + p_4^2). \quad (2.149)$$

For the construction of complex invariants using (2.149) for co-efficients, I_1 and I_2 are given by

$$I_1 = 4\rho_2(x_1^3p_3 + x_1p_3^3 + x_2^3p_4 + x_2p_4^3) + 2\rho_1(x_1^4 + x_2^4 - p_3^4 - p_4^4) - 2\rho_2(x_1^2 + x_2^2 + p_3^2 + p_4^2)$$
$$-2(p_1x_3 + p_4x_4)\{\beta_1(x_1 + x_2) + \beta_2(p_3 + p_4)\} + (p_1^2 + p_2^2 - x_3^2 - x_4^2)\{-\beta_2(x_1 + x_2) + \beta_1(p_3 + p_4)\}$$
$$+(p_1p_2 - x_3x_4)\{-\rho_2(x_1 + x_2) + \rho_1(p_3 + p_4)\} - (p_1x_4 + p_2x_3)\{\rho_1(x_1 + x_2) + \rho_2(p_3 + p_4)\}, \quad (2.150)$$

$$I_2 = -4\rho_1(x_1^3p_3 + x_1p_3^3 + x_2^3p_4 + x_2p_4^3) - 2\rho_2(x_1^4 + x_2^4 - p_3^4 - p_4^4) + 2\rho_1(x_1^2 + x_2^2 + p_3^2 + p_4^2)$$
$$+2(p_1x_3 + p_4x_4)\{-\beta_2(x_1 + x_2) + \beta_1(p_3 + p_4)\} + (p_1^2 + p_2^2 - x_3^2 - x_4^2)\{\beta_1(x_1 + x_2) + \beta_2(p_3 + p_4)\}$$
$$+(p_1p_2 - x_3x_4)\{\rho_1(x_1 + x_2) + \rho_2(p_3 + p_4)\} + (p_1x_4 + p_2x_3)\{-\rho_2(x_1 + x_2) + \rho_1(p_3 + p_4)\}. \quad (2.151)$$

Finally the complex invariant $I = I_1 + iI_2$ can be written as

$$I = 4b(x_1^3p_3 + x_1p_3^3 + x_2^3p_4 + x_2p_4^3) - 2b(x_1^4 + x_2^4 - p_3^4 - p_4^4) - 2b(x_1^2 + x_2^2 + p_3^2 + p_4^2)$$
$$+(p_1^2 - x_3^2 + 2ip_1x_3 + p_2^2 - x_4^2 + 2ip_4x_4)b[(x_1 + x_2) - i(p_3 + p_4)]$$
$$+(x_1 + x_2) - i(p_3 + p_4)e(p_1p_2 - x_3x_4 + ip_1x_4 + ip_2x_3). \quad (2.152)$$

In real and complex coordinate it is written as

$$I = \frac{w^2b}{3}(x^*x^3 + y^3y^*) + b(xx^* + yy^*) + \frac{b}{2}(x^* + y^*)(p_x^{*2} + p_y^{*2}) + e(x^* + y^*)p_xp_y.$$

2.3.5 Hamiltonian systems with quartic potential

This has indeed been found to be the case for several molecules, for example, tetrahydrofuran,, cyclopentane, and cyclobutanone [11]. These are generally five-membered ring compounds and a large quartic anharmonicity is expected for these vibrations. However, the determination of such two-dimensional anharmonic potentials from a spectrum is much more difficult than the corresponding problem in one dimension. Constructing complex invariant for above cases here, we first consider the case of $\mathcal{PT}$-symmetric Hamiltonian systems with in the frame work of rationalization method. Note that Hamiltonian for complex quartic potential is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\delta_1(x^2 + y^2) + \frac{1}{4}\delta_2(x^4 + y^4). \quad (2.153)$$

The complex version of (2.153), the $\mathcal{PT}$-symmetric one come by by using (2.71) in (2.153), as $H = H_1 + iH_2$ with

$$H_1 = \frac{1}{2}(p_1^2 + p_2^2 - x_3^2 - x_4^2) + \frac{1}{2}\delta_1(x_1^2 + x_2^2 - p_3^2 - p_4^2) + \frac{1}{4}\delta_2(x_1^4 + p_3^4 + x_2^4 + p_4^4 - 6x_1^2p_3^2 - 6x_2^2p_4^2);$$
$$H_2 = p_1x_3 + p_2x_4 + \delta_1(x_1p_3 + x_2p_4) + \delta_2(p_3x_1^3 - x_1p_3^3 + x_2^3p_4 - x_2p_4^2). \quad (2.154)$$

The above systems give access complex invariant I in the form

$$I = a_{01}(x,y) + a_{02}(x,y)(p_x^2 + p_y^2) + a_{11}(x,y)p_xp_y. \tag{2.155}$$

Substitution of (2.154), (2.155) in (2.78) yields the resultant expression which can be rationalized with respect to the power of p_1, x_3, p_2, x_4 and their combinations to give the following set of twelve coupled partial differential equations. Solutions to these PDE gives rise to forms of coefficient functions as

$$a_{01r} = -\frac{1}{3}\delta_1\beta_1(p_3^3 + p_4^3) - \frac{1}{3}\beta_2\delta_1(x_1^3 + x_2^3) - \beta_2\delta_1(x_1p_3^2 + x_2p_4^2) - \beta_1\delta_1(x_2^2p_4 + x_1^2p_3)$$
$$+\beta_2\delta_2(x_1p_3^4 + x_2p_4^4) - \beta_1\delta_1(x_1^4p_3 + x_2^4p_4) + \frac{1}{5}\delta_2\beta_1(p_3^5 + p_4^5) - \frac{1}{5}\beta_2\delta_2(x_1^5 + x_2^5);$$

$$a_{01i} = \frac{1}{3}\delta_1\beta_1(x_1^3 + x_2^3) - \frac{1}{3}\beta_2\delta_1(p_3^3 + p_4^3) + \beta_1\delta_1(x_1p_3^2 + x_2p_4^2) - \beta_2\delta_1(x_2^2p_4 + x_1^2p_3)$$
$$+\beta_2\delta_2(x_1p_3^4 + x_2p_4^4) - \beta_2\delta_1(x_1^4p_3 + x_2^4p_4) + \frac{1}{5}\delta_2\beta_2(p_3^5 + p_4^5) + \frac{1}{5}\beta_1\delta_2(x_1^5 + x_2^5). \tag{2.156}$$

For the construction of complex invariants using the results (2.156) for co-efficients, one can obtain the complex invariant $I = I_1 + iI_2$ that can be written as

$$I = \frac{b\delta_1}{6}[x^*(3x^2 - x^*) + y^*(3y^2 - y^*)] + \frac{b\delta_2}{20}[(5x^4x^* - x^{*5}) + (5y^4y^*y^{*5})]$$
$$+\frac{b}{2}[(x^* + y^*)(p_x^{*2} + p_y^{*2})] + e(x^* + y^*)p_xp_y.$$

2.3.6 Complex cubic potential

Spectral analysis of the complex cubic oscillator has been carried out by many researchers [8]. Using the exact semiclassical analysis, They studied the spectrum of a one-parameter family of complex cubic oscillators. Hamiltonian for such complex cubic potential is

$$H = (p_x^2 + p_y^2) + \delta_1(ix + iy) + \delta_2[(ix)^2 + (iy)^2] + \delta_3[(ix)^3 + (iy)^3]. \tag{2.157}$$

using (2.71) in (2.157), as gives us

$$H_1 = (p_1^2 + p_2^2 - x_3^2 - x_4^2) - \delta_1(p_3 + p_4) + \delta_2(p_3^2 + p_4^2 - x_1^2 - x_2^2) - \delta_3(p_3^3 + p_4^3 + 3x_1^2p_3 + 3x_2^2p_4);$$
$$H_2 = 2p_1x_3 + 2p_2x_4 + \delta_1(x_1 + x_2) - 2\delta_2(x_1p_3 + x_2p_4) - \delta_3(x_1^3 + x_2^3 + 3x_1p_3^2 + 3x_2p_4^2). \tag{2.158}$$

The systems may give the nod for complex invariant I in the form

$$I = a_{01}(x,y) + a_{02}(x,y)(p_x^2 + p_y^2) + a_{11}(x,y)p_xp_y \tag{2.159}$$

Substitution of (2.159), (in the form I_1, I_2) in (2.78) yields the expression and rationalization of the resultant expressions gives rise to forms of coefficient functions as

$$a_{01r} = \beta_1\delta_3(x_1^4 + x_2^4 - p_3^4 - p_3^4) - 2\beta_2\delta_3(x_1p_3^3 + x_1^3p_3 + x_2p_4^3 + x_2^3p_4) + 2\delta_2\beta_2(x_1p_3^2 + x_2p_4^2)$$
$$+2\delta_2\beta_1(x_1^2p_3 + x_2^2p_4) - \beta_1\delta_1(x_1^2 + x_2^2 + p_3^2 + p_3^2) + \frac{2\delta_2}{3}(\beta_1(p_3^3 + p_4^3) + \beta_2(x_1^3 + x_2^3));$$

$$a_{01i} = \beta_1\delta_3(x_1^4 + x_2^4 - p_3^4 - p_3^4) - 2\beta_2\delta_3(x_1p_3^3 + x_1^3p_3 + x_2p_4^3 + x_2^3p_4) + 2\delta_2\beta_2(x_1p_3^2 + x_2p_4^2)$$
$$+2\delta_2\beta_1(x_1^2p_3 + x_2^2p_4) - \beta_1\delta_1(x_1^2 + x_2^2 + p_3^2 + p_3^2) + \frac{2\delta_2}{3}(\beta_1(p_3^3 + p_4^3) + \beta_2(x_1^3 + x_2^3)). \quad (2.160)$$

For the construction of complex invariants using the results (2.160) for co-efficients, complex invariant $I = I_1 + I_2$ can be written as

$$I = \frac{ib\delta_2}{3}[x^*(x^{*2} - 3x^2) + y^*(y^{*2} - y^2)] - b\delta_1(xx^* + yy^*) + \delta_3(x^*x^{*3} + y^*y^{*3})$$
$$+e(x^* + y^*)(p_x^{*2} + p_y^{*2}) + e(x^* + y^*)p_xp_y$$

2.3.7 Invariants corresponding to non-linear evolution equations

The Koerteweg de-Vries (KdV) equations has wide ranging applications in non-linear physics [45]. Invariant corresponding to KdV equations may be interesting, we employ here the above method to construct an invariant for a Hamiltonian derived from this equation. The KdV equations in two-dimensional real field $u(x, y, t)$ is given by

$$\frac{\partial u}{\partial t} + \bar{a}u\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right] + \eta\left[\frac{\partial^3 u}{\partial^3 x} + \frac{\partial^3 u}{\partial^3 y}\right]. \quad (2.161)$$

Corresponding to this equation, the Hamiltonian can be written as

$$H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{\gamma}{2}(x^2 + y^2) + \left(\frac{\bar{a}}{6}\right)(x^3 + y^3). \quad (2.162)$$

The $\mathcal{PT}$-symmetric Hamiltonian can be attain by using (2.71) in (2.162), as $H = H_1 + iH_2$ with

$$H_1 = \frac{1}{2}(p_1^2 - x_3^2 + p_2^2 - x_4^2) + \frac{\gamma}{2}(x_1^2 - p_3^2 - x_2^2 - p_4^2) + \frac{\bar{a}}{6}(x_1^3 + x_2^3 - 3x_1p_3^2 - 3x_2p_4^2);$$
$$H_2 = p_1x_3 + p_2x_4 - \gamma(x_1p_3 + x_2p_4) + \frac{\bar{a}}{6}(3x_1^2p_3 + 3x_2^2p_4 - p_3^3 - p_4^3). \quad (2.163)$$

permit the complex invariant I. Substitution of (2.163), and I in (2.78) yields the resultant expression, which can be rationalized with respect to the power of p_1, x_3, p_2, x_4 and their combinations to give the following set of twelve coupled partial differential equations. These PDE propels the forms of coefficient functions as

$$a_{01r} = a_{01xr} + a_{01yr} = \beta_2\gamma(x_1p_3^2 + x_2p_4^2) + \beta_1\gamma(x_1^2p_3 + x_2^2p_4) + \frac{1}{3}\beta_1\gamma(p_3^3 + p_4^3) + \frac{1}{3}\beta_2\gamma(x_1^3 + x_2^3)$$
$$-\frac{\bar{a}}{6}\beta_2(x_1^4 + x_2^4 - p_3^4 - p_4^4) - \frac{\bar{a}}{3}\beta_1(x_1p_3^3 + x_1^3p_3 + x_2p_4^3 + x_2^3p_4);$$

$$a_{01i} = a_{01xi} + a_{01yi} = -\beta_1\gamma(x_1p_3^2 + x_2p_4^2) + \beta_2\gamma(x_1^2p_3 + x_2^2p_4) + \frac{1}{3}\beta_2\gamma(p_3^3 + p_4^3) - \frac{1}{3}\beta_1\gamma(x_1^3 + x_2^3)$$
$$+\frac{\bar{a}}{6}\beta_1(x_1^4 + x_2^4 - p_3^4 - p_4^4) - \frac{\bar{a}}{3}\beta_2(x_1p_3^3 + x_1^3p_3 + x_2p_4^3 + x_2^3p_4). \quad (2.164)$$

corresponding to this complex invariants can be written as

$$I = \frac{b\gamma}{6}[x^*(3x^2 - x^{*2}) + y^*(3y^2 - y^{*2})] + \frac{b\bar{a}}{6}(x^3x^* + y^3y^*) + \frac{e}{2}(x^* + y^*)(p_x^{*2} + p_y^{*2}) + b(x^* + y^*)p_xp_y$$

which conforms the invariance condition for integrability with view of poisson bracket.

2.3.8 Shifted harmonic oscillator complex x-plane

This type of oscillator has been discussed by [10] in context of neutron scattering. They studied the spin-orbit coupling term in shell model by neutron scattering using complex harmonic oscillator potential. For complex TD harmonic oscillator system, there exist a complex invariant, $u = ln(p + im\omega x) - i\omega t$ (as cited in [2]), has already been known in literature. Hamiltonian for shifted harmonic oscillator complex x-plane is given as

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}w^2(x + y + \frac{1}{2}i\gamma)^2 \tag{2.165}$$

where ω and γ are real constants.This form of H, after appropriate scaling of x and p (with $\omega = 1$), it is rewritten as,

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + i\gamma(x + y)$$

using (2.71) in (2.165) as

$$\begin{aligned} H_1 &= \frac{1}{2}(p_1^2 + p_2^2 + x_1^2 + x_2^2 - p_3^2 - p_4^2 - x_3^2 - x_4^2) - \gamma(x_1 p_4 + x_2 p_3); \\ H_2 &= p_1 x_3 + p_2 x_4 + x_1 p_3 + x_2 p_4 + \gamma(x_1 x_2 - p_3 p_4). \end{aligned} \tag{2.166}$$

Analyticity property of H allows that H_2 is the second constant of the motion of the system in the sense that $[H_1, H_2] = 0$, and H_1 and H_2 are linearly independent with respect to the canonical pairs $(x_1, p_1), (x_2, p_2), (x_3, p_3)$ and (x_4, p_4). The above systems take in the complex invariant I in the form

$$I = a_{01}(x, y) + a_{02}(x, y)(p_x^2 + p_y^2) + a_{11}(x, y)p_x p_y \tag{2.167}$$

Substituting (2.166), (2.167) in (2.78) and rationalizing the resultant expression and their combinations to give the following set of twelve coupled partial differential equations: Following the prescription outlined in previous example the coefficient functions which gives rise to forms

$$\begin{aligned} a_{01r} &= -\frac{4}{3}[\beta_2(x_1^3 + x_2^3) + \beta_1(p_3^3 + p_4^3) - \beta_1\gamma(x_1^2 + x_2^2 + p_3^2 + p_4^2) + c_1(x_1 + x_2) + d_2(p_3 + p_4); \\ a_{01i} &= \frac{4}{3}[\beta_1(x_1^3 + x_2^3) - \beta_2(p_3^3 + p_4^3)] - \beta_2\gamma(x_1^2 + x_2^2 + p_3^2 + p_4^2) + d_1(x_1 + x_2) - c_1(p_3 + p_4). \end{aligned} \tag{2.168}$$

one can obtain the complex invariant I using (2.168),

$$\begin{aligned} I &= \frac{b}{3}[x^*(3x^2 + x^*) + y^*(3y^2 + y^*)] + ib\gamma(xx^* + yy^*) + \sigma(x^* + y^*) \\ &+ b[(x^* + y^*) + \frac{\sigma_0}{2w^2}](p_x^{*2} + p_y^{*2}) + e(x^* + y^*)p_x p_y. \end{aligned}$$

which conforms the invariance condition for integrability with view of poisson bracket.

2.4 Construction of fourth order invariants

The problem of construction of invariants (other than the total energy, which is called second constant of motion) for TID systems has been investigated by many workers [2, 3, 4, 5, 13, 14]. In past a large number of dynamical systems have been studied using approximate method or perturbation method [15] and accordingly one deals the approximate invariants for the system. In some cases, no doubt the system is found to be integrable just by an accident by Hénon *et al* e.g. Toda lattice case [15]. These studies, however, facilitates the solution of nonlinear differential equations. The construction of the additional invariants (if possible) for a dynamical system help a lot in understanding the dynamics of the corresponding system.

We are here interested in to find fourth order invariants of two dimensional classical systems, so consider such system described by the Lagrangian

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - V(x_1, x_2),$$

the associated equation of motion given by

$$\ddot{x_1} = -\frac{\partial V}{\partial x_1}; \qquad \ddot{x_2} = -\frac{\partial V}{\partial x_2},$$

where $V = V(x_1, x_2, x_3)$.

Their studies were, however, restricted to the invariants of first or second order in momenta. During the last few decades or so, there have been considerable interest in the study of TID and TD classical integrable systems in one and two dimensions. Several authors [5, 13, 16, 17, 18] have studied construction of invariants and its various aspects in many folds. Although, there have been several efforts [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31] in recent years to construct second order invariants by various methods in one and two dimensions, however not much efforts have been made to obtaining the invariants of higher order in three dimensions of such systems. In some cases, no doubt the system is found to be integrable just by accident. For interesting reviews on the subject, we refer to the works of Hietarinta [16], Hall [18], Holt [24], and R.S.Kaushal [4]. It may be mentioned that a large number of dynamical systems in the literature have been studied using the so called approximate methods or perturbation methods [18, 32], and accordingly one deals the approximate invariants for the system. These studies, however, facilitates the solutions of nonlinear differential equations.

The construction of invariants of some systems either have played or expected to play an important role in the domains of a variety of fields like, laser physics [32] plasma physics [33], molecular physics [34], field theories [35], astrophysics [36], etc. The invariants of a dynamical system, if they exist, and become available could be of great help in providing a physical insight into the detailed properties of the system.

The construction of an invariant of a system could be helpful in reducing some nonlinear dynamical problems to a quadrature [1, 37]. The importance of these invariants clearly offers a motivation to look for the systems that can generate these structures in an unambiguous manner. Here in this section we have extended the rationalization method for construction of fourth order invariants of a number of three dimensional classical systems. There exist, at present, no general method for testing the integrability of a given system, however, the Painlevé method [1] detects the integrability of a dynamical system with the use of singularity analysis and direct calculation of the second integral of motion. Grammaticos *et al.* [25] discussed a method of constructing N-dimensional integrable system starting from two dimensional one. They further carried out the singularity analysis of the equation of motion, which led them to system exhibiting the Painlevé property i.e.the only movable singularities of the solutions in the complex time plane were assumed to be pole type. These results are discussed for different cases of N-dimensional system by Lakshmanan and Sahadevan [39]. In the recent past, Holt [30] introduced a procedure which essentially has bearing in perturbation theory of Mc.Namara and Whiteman [32] who obtained the third order invariants for a number of integrable systems. Dorrizi *et al.* [39] investigated the existence of integrable systems in three dimensions in which they reduced three dimensional system to two dimensional one, using cylindrical symmetry and solved quartic potentials. For the construction of invariants, a variety of techniques have been employed in the past with varying degree of success and domains of applicability [22, 23, 24, 25].

A classical Hamiltonian systems of n-degrees of freedom is said to be classical integrable if there are $(n-1)$ independent, well defined, global functions whose Poisson bracket with each other and with the Hamiltonian vanishes. Looking at the quantum integrability the above statement reads: a quantum Hamiltonian systems of n-degrees of freedom is said to be quantum integrable if there are $(n-1)$ independent, well defined, global operators, which commute with each other and with the Hamiltonian. A simple comparison between classical and quantum integrability can be easily made if one represent the quantum operators by c-number functions. It has been studied that the classical and quantum mechanics are not algebraic isomorphic, differences must be present somewhere if we use c-number functions in both. As a matter of fact the commutator will be the Moyal bracket [19], which in turn reduces to the Poisson bracket only when $\hbar \to 0$ in the expansion of sine function [20, 21]. Relationship between a classical invariant and its quantum counterpart in not one to one correspondence, however, some close correspondence may exist. In fact, in order to obtain the quantum invariant of a system from the corresponding classical one, the quantum corrections arising from the terms involving $\hbar$ in the expansion of the sine-function need to be incorporated. It has been observed that cubic and higher order invariants need the quantum corrections for the existence of quantum invariant [2]. In fact, after incorporating these corrections, a system becomes quantum integrable.

Now in this section we have extended the rationalization method for construction of fourth order invariant for two dimensional systems.

2.4.1 Method for fourth order I in two-dimension

We are interested in to find fourth order invariants of two dimensional classical systems [40, 41], so consider such system described by the Lagrangian

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - V(x_1, x_2), \tag{2.169}$$

the associated equation of motion given by

$$\ddot{x_1} = -\frac{\partial V}{\partial x_1}, \quad \ddot{x_2} = -\frac{\partial V}{\partial x_2}. \tag{2.170}$$

Let us assume that there exist a second constant of motion, fourth order in momenta, for the system of eq.(2.169)

$$I = f_1 p_x^4 + f_2 p_x^3 p_y + f_3 p_x^2 p_y^2 + f_4 p_x p_y^3 + f_5 p_y^4 + f_6 p_x^2 + f_7 p_x p_y + f_8 p_y^2 + f_9. \tag{2.171}$$

Where $f_1, f_2, ...f_9$ are the functions of x and y only and $p_x, \ p_y$ are momenta in two dimension. The invariant I implies $dI/dt = 0$ and using eq.(2.171) we get

$$\begin{aligned}
&f_{1x} p_x^5 + f_{1y} p_x^4 p_y + 4 f_1 p_x^3 \dot{p_x} + f_{2x} p_x^4 p_y + f_{2y} p_x^3 p_y^2 + f_2(3 p_x^2 \dot{p_x} p_y + p_x^3 \dot{p_y}) + f_{3x} p_x^3 p_y^2 \\
&+ f_{3y} p_x^2 p_y^3 + f_3(2 p_x \dot{p_x} p_y^2 + 2 p_y \dot{p_y} p_x^2) + f_{4x} p_x^2 p_y^3 + f_{4y} p_x p_y^4 \\
&+ f_4(\dot{p_x} p_y^3 + 3 p_y^2 \dot{p_y} p_x) + f_{5x} p_x p_y^4 + f_{5y} p_y^5 + 4 f_5 \dot{p_y} p_y^3 + f_{6x} p_x^3 \\
&+ f_{6y} p_x^2 p_y + 2 f_6 \dot{p_x} p_x + f_{7x} p_x^2 p_y + f_{7y} p_y^2 p_x + f_7(\dot{p_x} p_y + \dot{p_y} p_x) \\
&+ f_{8x} p_x p_y^{=2} + f_{8y} p_y^3 + 2 f_8 \dot{p_y} p_y + f_9 p_x + f_9 p_y = 0.
\end{aligned}$$

Now after comparing the coefficient of powers of p_x, p_y and their respective products we get the following set of equations.

$$f_{1x} = 0, \tag{2.172}$$

$$f_{1y} + f_{2x} = 0, \tag{2.173}$$

$$f_{2y} + f_{3x} = 0, \tag{2.174}$$

$$f_{3y} + f_{4x} = 0, \tag{2.175}$$

$$f_{4y} + f_{5x} = 0, \tag{2.176}$$

$$f_{5y} = 0, \tag{2.177}$$

$$4f_1\ddot{x} + f_2\ddot{y} + f_{6x} = 0, \tag{2.178}$$

$$3f_2\ddot{x} + 2f_3\ddot{y} + f_{6y} + f_{7y} = 0, \tag{2.179}$$

$$2f_3\ddot{x} + 3f_4\ddot{y} + f_{7y} + f_{8x} = 0, \tag{2.180}$$

$$f_4\ddot{x} + 4f_5\ddot{y} + f_{8y} = 0, \tag{2.181}$$

$$2f_6\ddot{x} + f_7\ddot{y} + f_{9x} = 0, \tag{2.182}$$

$$f_7\ddot{x} + 2f_8\ddot{y} + f_{9y} = 0, \tag{2.183}$$

where dot represent differentiation w.r.t. 't'.

2.4.2 Determination of coefficients

First let us find out the solution of potential independent equations (2.172)-(4.7). Now differentiating eq.(2.173) w.r.t. y and eq.(2.174) w.r.t. x and subtract then, we get

$$f_{1yy} + f_{3xx} = 0, \tag{2.184}$$

Differentiating eq.(4.14) w.r.t. y and double differentiating eq.(2.175) w.r.t. x and add, we get

$$f_{1yyy} + f_{4xxx} = 0, \tag{2.185}$$

Differentiating eq.(4.15) w.r.t. y and differentiating eq.(2.176) three times w.r.t. x and subtract, we get

$$f_{1yyyy} - f_{5xxxx} = 0, \tag{2.186}$$

or

$$f_{1yyyy} = f_{5xxxx} = \epsilon_0, \tag{2.187}$$

Now we get the solution of f_1 from eq.(4.17) as

$$f_1 = \epsilon_0 y^4 + \epsilon_1 y^3 + \epsilon_2 y^2 + \epsilon_3 y + \epsilon_4. \tag{2.188}$$

Now to find the solution of f_2 differentiate eq.(2.174) w.r.t. y and eq.(2.175) w.r.t. x and subtract then we get

$$f_{2yy} = f_{4xx}, \tag{2.189}$$

Again differentiate eq.(2.189) w.r.t. y and differentiate eq.(2.176) w.r.t. x and add then next equation becomes

$$f_{2yyy} = -f_{5xxx}, \tag{2.190}$$

Let

$$f_{2yyy} = -f_{5xxx} = \eta_0, \tag{2.191}$$

So form eq.(2.190) and eq.(2.173) we get general solution for f_2 as

$$f_2 = -(4\epsilon_0 y^3 + 3\epsilon_1 y^2 + 2\epsilon_2 y + \epsilon_3)x + \eta_0 y^3 + \eta_1 y^2 + \eta_2 y + \eta_3, \tag{2.192}$$

For the solution of f_3 differentiate eq.(2.175) w.r.t. y and differentiate eq.(2.176) w.r.t. x and subtract we get

$$f_{3yy} = f_{5xx} = \sigma_0. \tag{2.193}$$

Hence from eq. (2.193) and eq.(2.174) we get general solution of f_3 as

$$f_3 = (6\epsilon_0 y^2 + 3\epsilon_1 y + \epsilon_2 +)x^2 - (3\eta_0 y^2 + 2\eta_1 y + \eta_2)x + \sigma_0 y^2 + \sigma_1 y + \sigma_2. \tag{2.194}$$

For the solution of f_4, differentiate eq.(2.176) w.r.t. y and differentiate eq.(4.7) w.r.t. x and subtract we get

$$f_{4yy} = 0. \tag{2.195}$$

Hence from eq.(2.175), (2.189) and (2.195) we get general solution of f_4 as

$$f_4 = -(4\epsilon_0 y + \epsilon_1)x^3 + (3\eta_0 y + \eta_1)x^2 - (2\sigma_0 y + \sigma_1) + \theta_0 y + \theta_1. \tag{2.196}$$

For the solution of f_5, we use eq.(2.174), (4.17), (2.191) and (2.193) then we get general solution of f_5 as

$$f_5 = \epsilon_0 x^4 - \eta_0 x^3 + \sigma_0 x^2 - \theta_0 x + \theta. \tag{2.197}$$

Now from eqs.(4.8),-(4.11), we get first compatibility condition as

$$(f_4\ddot{x} + 4f_5\ddot{y})_{xxx} - (2f_3\ddot{x} + 3f_4\ddot{y})_{xxy} + (3f_2\ddot{x} + 2f_3\ddot{y})_{xyy} - (4f_1\ddot{x} + f_2\ddot{y})_{yyy} = 0. \tag{2.198}$$

The second compatibility condition or "potential equation" can be found using eqs.(4.12) and (4.13) as

$$(2f_6\ddot{x} + f_7\ddot{y})_y - (f_7\ddot{x} + 2f_8\ddot{y})_x = 0, \tag{2.199}$$

The solution of these potential equations would likely provide the integrable systems. For a given form of potential $V(x, y)$ all unknown integration constants can be determined by rationalizing the potential equation, subsequently, the determination of other coefficients f_6 to f_9 from eq.(4.8) to (4.13). On substitution these coefficients in eq.(2.171) which would lead to the final form of the second constant of motion. This constant of motion must commute with Hamiltonian. Here, in the next section, we find out the constant of motion for a potential which is separable in addition.

(1) Example One.

In this section we use the potential eq.(2.198) to construct the second constant of motion for a given potential. To demonstrate the method we take the potential which is separable in addition and the form of potential is like,

$$V(x,y) = x^n + y^m, \tag{2.200}$$

where m and n are the numbers and Hamiltonian for this potential is given by

$$H(x,y,p_x,p_y) = \frac{1}{2}(p_x^2 + p_y^{2)}) + x^n + y^m. \tag{2.201}$$

Substitute this form of potential in the potential equation i.e. the first compatibility condition (2.198). The compatibility condition satisfy only for $m = n = -2$

$$\epsilon_3 = \eta_0 = \eta_2 = \eta_3 = \sigma_1 = \theta_0 = \theta_1 = 0, \tag{2.202}$$

Now coefficients f_6, f_7, f_8, and f_9 can be determined from the eq.(4.8) - (4.13).

$$\begin{aligned} f_6 &= 4\epsilon_0 x^{-2}y^4 + \epsilon_1 y^3 x^{-2} + 2\epsilon_2 x^{-2}y^2 + 4\epsilon_4 x^{-2} \\ &\quad - 2\eta_1 x y^{-1} + 4\epsilon_0 x^2 + 2\epsilon_1 x^2 y^{-1} + \epsilon_2 x^2 y^{-2} + K_1'(y) \end{aligned} \tag{2.203}$$

$$\begin{aligned} f_7 &= -4\epsilon_0 x^{-1}y^3 - \epsilon_1 x^{-1}y^2 - 4\epsilon_2 x^{-1}y - 4\epsilon_0 x^3 y^{-1} - \epsilon_1 x^3 y^{-2} \\ &\quad + 3\eta_1 x^2 y^{-2} + 3\eta_1 x^{-2}y^2 - 4\sigma_0 x y^{-1}, \end{aligned} \tag{2.204}$$

$$f_8 = 4\epsilon_0 y^2 + 4\epsilon_0 x^4 y^{-2} + 2\eta_1 x^{-1} y + \epsilon_1 y + 2\sigma_0 x^2 y^{-2} + 4\theta_2 y^{-2} + K_2'(x), \tag{2.205}$$

and

$$\begin{aligned} f_9 &= 4\epsilon_0 x^{-4}y^4 + 4\epsilon_0 x^4 y^{-4} - \epsilon_1 x^{-4}y^3 + 4\epsilon_2 x^{-4}y^2 + \epsilon_1 x^4 y^{-5} + 4\epsilon_4 x^{-4} - \eta_1 x^{-1}y^{-1} \\ &+ 4\sigma_2 x^2 y^{-6} - 4\sigma_2 x^{-2}y^2 + 4\sigma_0 x^2 y^{-4} - 3\eta_1 x^3 y^{-5} - 3\eta_1 x^{-5}y^3. \end{aligned} \tag{2.206}$$

But to satisfy the second compatibility condition some more integration constants also vanish i.e.

$$\epsilon_1 = \epsilon_4 = \eta_1 = \sigma_2 = \theta_2 = K_1'(y) = K_2'(x) = 0 \tag{2.207}$$

Therefore, for the present case, the solution for f_1 - f_9 are written as

$$f_1 = \epsilon_2 y^2, \tag{2.208}$$

$$f_2 = -2\epsilon_2 yx, \tag{2.209}$$

$$f_3 = \epsilon_2 x^2 + \sigma_0 y^2, \tag{2.210}$$

$$f_4 = -2\sigma_0 yx, \tag{2.211}$$

$$f_5 = \sigma_0 x^2, \tag{2.212}$$

$$f_6 = 4\epsilon_2 y^2 x^{-2} + 2\epsilon_2 y^{-2} x^2, \tag{2.213}$$

$$f_7 = -4\epsilon_2 yx^{-1} - 4\sigma_0 y^{-1} x, \tag{2.214}$$

$$f_8 = 2\sigma_0 y^2 x^{-2} + 4\sigma_0 y^{-2} x^2, \tag{2.215}$$

$$f_9 = 4\epsilon_2 y^2 x^{-4} + 4\sigma_0 y^{-4} x^2. \tag{2.216}$$

After putting values of these coefficients in eq.(2.171) we get the final form of the second constant of motion for a dynamical system with a given potential form $V(x, y) = x^{-2} + y^{-2}$. The final form of the invariant is

$$\begin{aligned} I &= \epsilon_2 y^2 p_x^4 - 2\epsilon_2 yxp_x^3 p_y + \epsilon_2 x^2 p_x^2 p_y^2 + \sigma_0 y^2 p_x^2 p_y^2 - 2\sigma_0 yxp_x p_y^3 + \sigma_0 x^2 p_y^4 + 4\epsilon_2 y^2 x^{-2} p_x^2 \\ &+ 2\epsilon_2 y^{-2} x^2 p_x^2 - 4\epsilon_2 yx^{-1} p_x p_y - 4\sigma_0 y^{-1} xp_x p_y + 2\sigma_0 y^2 x^{-2} p_y^2 + 4\sigma_0 y^{-2} x^2 p_y^2 + 4\epsilon_2 y^2 x^{-4} + 4\sigma_0 y^{-4} x^2. \end{aligned} \tag{2.217}$$

(2) Second example

Let us consider a potential which is a combination of harmonic plus inverse harmonic terms, so mathematically one can write

$$V(x, y) = x^2 + y^2 + x^{-2} + y^{-2}. \tag{2.218}$$

Before proceeding towards invariant construction of the potential, a few remarks regarding to this potential are in order: this potential has been studied widely on both quantum and classical level in past. The terms x^{-2} and y^{-2} are corresponding to a nonlinear centrifugal force and appear in many integrable systems, including the celebrated Calogero-Sutherland-Moser [42, 43, 44] many body Hamiltonian. This potential also represents a harmonic oscillator in the presence of repulsive dipole potential. Now the corresponding Hamiltonian is given by

$$H(p_x,p_y,x,y)=\frac{1}{2}(p_x^2+p_y^2)+x^2+y^2+x^{-2}+y^{-2}. \tag{2.219}$$

Substituting this form of potential in eq.(2.198) i.e. compatibility condition and to satisfy this condition some of the integration constants have to be vanished, these are

$$\epsilon_1=\epsilon_3=\eta_0=\eta_2=\eta_3=\sigma_0==\sigma_1=\theta_0=\theta_1=0, \tag{2.220}$$

Now the coefficients f_6, f_7, f_8, and f_9 can be calculated and these are

$$\begin{aligned} f_6 \quad &= 2\epsilon_2x^2y^2+4\epsilon_4x^2+4\epsilon_0x^{-2}y^4+4\epsilon_2x^{-2}y^2+2\eta_1xy^3+4\epsilon_0x^2 \\ &+2\epsilon_2x^2y^{-2}-2\eta_1xy^{-1}+4\epsilon_4x^{-2}+K_1(y), \end{aligned} \tag{2.221}$$

$$\begin{aligned} f_7 \quad &= 4\epsilon_2x^3y-8\epsilon_0x^{-1}y^3-4\epsilon_2x^{-1}y-8\epsilon_0x^3y^{-1}-4\eta_1x^2y^2 \\ &+3\eta_1x^{-2}y^2+3\eta_1x^2y^{-2}, \end{aligned} \tag{2.222}$$

$$f_8=4\epsilon_0y^2+4\epsilon_0x^4y^{-2}+2\eta_1x^3y-2\eta_1x^{-1}y+K_2(x), \tag{2.223}$$

and

$$f_9=4\epsilon_0x^{-4}y^4+4\epsilon_0x^4y^{-4}+4\epsilon_4x^4+4\epsilon_4x^{-4}. \tag{2.224}$$

But to satisfy second compatibility condition some more coefficients also vanish and these coefficients are

$$\epsilon_4=\eta_1=\sigma 2=\theta_2=K_1(x)=K_2(y)=0. \tag{2.225}$$

Then the reduced form of coefficients are

$$f_1=\epsilon_0y^4, \tag{2.226}$$

$$f_2=-4\epsilon_0y^3x, \tag{2.227}$$

$$f_3=6\epsilon_0y^2x^2, \tag{2.228}$$

$$f_4=-4\epsilon_0yx^3, \tag{2.229}$$

$$f_5=\epsilon_0x^4, \tag{2.230}$$

$$f_6=4\epsilon_0y^4x^{-2}, \tag{2.231}$$

$$f_7 = -8\epsilon_0 y^3 x^{-1} - 8\epsilon_0 y^{-1} x^3, \tag{2.232}$$

$$f_8 = 4\epsilon_0 y^2 + 4\epsilon_0 y^{-2} x^4, \tag{2.233}$$

$$f_9 = 4\epsilon_0 y^4 x^{-4} + 4\epsilon_0 y^{-4} x^4. \tag{2.234}$$

After putting these values of coefficients in eq.(2.171), we get the final form of the second invariant for the given dynamical system as

$$\begin{aligned} I &= \epsilon_0 y^4 p_x^4 - 4\epsilon_0 y^3 x p_x^3 p_y + 6\epsilon_0 y^2 x^2 p_x^2 p_y^2 - 4\epsilon_0 y x^3 p_x p_y^3 + \epsilon_0 x^4 p_y^4 + 4\epsilon_0 y^4 x^{-2} p_x^2 - 8\epsilon_0 y^3 x^{-1} p_x p_y \\ &- 8\epsilon_0 y^{-1} x^3 p_x p_y + 4\epsilon_0 y^2 p_y^2 + 4\epsilon_0 y^{-2} x^4 p_y^2 + 4\epsilon_0 y^4 x^{-4} + 4\epsilon_0 y^{-4} x^4. \end{aligned} \tag{2.235}$$

In this section we have constructed fourth order invariants for a class of two dimensional potentials with the help of potential equation eq.(2.198) and (2.199). These invariants also commutes with given Hamiltonians. In the present work, we rather revisit the construction of second constant of motion for TID classical systems in two-dimensions using rationalization method. Using Moyal's bracket, we obtain quantum integrable systems with quantum corrections to the corresponding classical invariants.

In the fifth chapter we will construct the higher order invariants for quantum cases for two dimensional time independent systems.

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Chapter 3

Lie-algebraic Method:Invariants for Classical Systems in ECPS

For obtaining invariants of a variety of time dependent (TD) systems, researchers used the closure property of dynamical Lie-algebra generated by phase space functions [1, 2, 3]. Though this method is easy to extend to the quantum domain [4], it turns out to be relatively more efficient for the TD systems. In recent years, a considerable progress has been made to develop the theory of complex dynamical invariants for both TD and TID classical systems [5, 6, 7]. First the existence and subsequently the construction, if possible, of these additional invariants for a dynamical system help a lot in understanding the detailed properties of the corresponding system.

For the construction of invariants a variety of techniques have been employed in the past with varying degrees of success and domains of applicability. In this context, while more concerted efforts have been made for the TID systems, several methods have been used for the systems involving explicit TD in two dimensions. Sometimes from the point of view of mathematical abstraction a TD n-dimensional Hamiltonian system is considered as an $(n+1)$-dimensional Hamiltonian system in which the time appears as a new canonical coordinate [8]. However, for the practical applications of the theoretical framework for TD systems a separate account of its time variable is inevitable in spite of the fact that in given dimensions the time dependence of the system (if it exists) makes its study more complicated than the corresponding TID situation. Moreover, the study of TD systems in higher dimensions itself becomes rather involved and mainly for this reason such systems are not explored to the same extent as the TID ones. An additional advantage of this Lie algebraic method is that the transition to the corresponding quantum mechanical system is easier and it becomes straightforward.

3.1 Construction of complex invariants in one dimensions

Here as for the real Hamiltonian systems one can expands the Hamiltonian $H(x,p,t)$ in to a complete set of ECPS as $H(x_1,p_2,p_1,x_2,t)$. The $H(x,p,t)$ of the system can expressed as

$$H = \sum_n h_n(t)\Gamma_n(x_1,x_2,p_1,p_2,t), \text{where } \Gamma \text{ is new phase space function} \tag{3.1}$$

The coefficients in the expansion now are however the complex functions of the real parameter t. The fact that the invariant I is also a member of this set and allows one to look for the closure of the Lie algebra generated by the Γ's with respect to the Poisson brackets. Lie-algebraic approach [2, 3] at the classical level has been used to construct complex integrals for the complex dynamical systems. Though this method is easy to extend to the quantum domain, it turns out to be relatively more efficient for the TD systems. For using Lie-algebraic approach recall this method, which has been briefly described in section 1.5 of chapter one.

Complex invariants in one-dimensions

Most of the studies on this front carried out only for real one-dimensional systems. Here, in the present work we carried out the extended phase plane approach in one-dimensions with a view to obtain exact complex integrals of classical dynamical systems. Lie-algebraic method is explored for such constructions as this has been widely used in literature for the construction of exact and real invariants for a variety of classical dynamical systems [3] and can easily be extended for its corresponding quantum systems also.

The Method for one-dimensions

As we have already discussed in section 3.2 of chapter 3 that a one-dimensional real phase space (x,p), can be transform into a complex space (x_1,x_2,p_1,p_2), by using eq.(2.5) as

$$x = x_1 + ip_2; \qquad p = p_1 + ix_2. \tag{3.2}$$

Therefore, the Hamiltonian $H(x,p)$ of a one dimensional system in complex space can be expressed, using eq.(3.2), as $H = H_1 + iH_2$. Clearly, from eq.(3.2) we get

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_2}, \qquad \frac{\partial}{\partial p} = \frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_2}. \tag{3.3}$$

The Hamilton's equations of motion for complex H in eq.(3.2) can be written as

$$\begin{aligned} \dot{x}_1 &= \left(\frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_2}\right); \qquad & \dot{p}_2 &= \left(\frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_2}\right); \\ \dot{p}_1 &= -\left(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_2}\right); \qquad & \dot{x}_2 &= -\left(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_2}\right). \end{aligned} \tag{3.4}$$

If the Hamiltonian H in eq.(3.4) is to be analytic function of complex variables, then H_1 and H_2 satisfy the Cauchy-Riemann conditions [16] and after employing such analyticity conditions, eq.(**??**) becomes

$$\dot{x}_1 = 2\frac{\partial H_1}{\partial p_1}; \qquad \dot{p}_1 = -2\frac{\partial H_1}{\partial x_1}; \quad \dot{x}_2 = 2\frac{\partial H_1}{\partial p_2}; \qquad \dot{p}_2 = -2\frac{\partial H_1}{\partial x_2}. \tag{3.5}$$

Note that$(x_1, p_1), (x_2, p_2)$ constitute canonical pairs. Now consider a complex phase space function $I(x, p, t)$ as $I = I_1 + iI_2$. It is quite interesting and important in the study of integrable system in the Liouville sense such as those associated with certain physical meaning equations. This requires function I to follow the invariance condition the time dependent system in complex phase space, as

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H] = 0, \tag{3.6}$$

where $[., .]$ is the Poisson bracket, which in view of the definition eq.(3.2) turns out as

$$[A, B]_{(x,p)} = [A, B]_{(x_1,p_1)} - i[A, B]_{(x_1,x_2)} - i[A, B]_{(p_2,p_1)} - [A, B]_{(p_2,x_2)}. \tag{3.7}$$

the computation of Poisson brackets and to satisfy the closure property of Lie algebra in case of complex Hamiltonian systems is a bit of tedious work.

3.1.1 Some examples

The described approach gives a flexible and effective tool for investigations of complex invariant. We will discuss some different examples for dynamical systems.

1. First Illustrative example

Consider a momentum dependent coupled harmonic oscillator in one-dimension [12], whose Hamiltonian is given by

$$H = \frac{1}{2}p^2 + \frac{1}{2}a_0(t)x^2 + a_1(t)xp \tag{3.8}$$

Here we demonstrate that the complex version of (3.8), namely the $\mathcal{PT}$-symmetric one obtained by using (3.2) in (3.8), the above Hamiltonian can be expressed as

$$H = \frac{1}{2}p_1^2 - \frac{1}{2}x_2^2 + ip_1x_2 + a_0(t)[\frac{1}{2}x_1^2 + ip_2x_1 - \frac{1}{2}p_2^2] + a_1(t)[x_1p_1 - p_2x_2 + ip_1p_2 + ix_1x_2] \tag{3.9}$$

$$= \sum_{m=1}^{10} h_m(t)\Gamma_m(x_1, x_2, p_1, p_2, t). \tag{3.10}$$

and the various Γ's and $h(t)'s$ for the above complex H are given as

$$\begin{array}{lllll} \Gamma_1 = \frac{1}{2}p_1^2; & \Gamma_2 = \frac{1}{2}x_2^2; & \Gamma_3 = p_1x_2; & \Gamma_4 = \frac{1}{2}x_1^2; & \Gamma_5 = \frac{1}{2}p_2^2; \\ \Gamma_6 = p_2x_1; & \Gamma_7 = x_1p_1; & \Gamma_8 = p_2x_2; & \Gamma_9 = p_1p_2; & \Gamma_{10} = x_1x_2; \end{array} \tag{3.11}$$

with

$$h_1 = -h_2 = 1;\ h_3 = i;\ h_4 = ih_5 = ia_0(t);\ h_6 = -h_7 = a_1(t) = -h_8;\quad h_9 = ia_1(t) = h_{10}. \tag{3.12}$$

The dynamical algebra in this case is not closed. To find closure property for the above system, we add four more phase space functions Γ_l's. The additional Γ_l's are as follow

$$\Gamma_{11} = p_1;\quad \Gamma_{12} = x_2;\quad \Gamma_{13} = x_1;\quad \Gamma_{14} = p_2;\qquad h_{11} = h_{12} = h_{13} = h_{14} = 0. \tag{3.13}$$

with corresponding $h_l(t) = 0$. Now in the light of Poisson bracket for complex systems, eq.(3.7), we get large number of nonvanishing Poisson brackets, namely

$$\begin{aligned}
&[\Gamma_1,\Gamma_4] = -\Gamma_7;\quad [\Gamma_1,\Gamma_5] = i\Gamma_9; [\Gamma_1,\Gamma_6] = -\Gamma_9 + i\Gamma_7; [\Gamma_1,\Gamma_7] = -2\Gamma_1; [\Gamma_1,\Gamma_8] = i\Gamma_3; [\Gamma_1,\Gamma_9] = 2i\Gamma_1;\\
&[\Gamma_1,\Gamma_{10}] = -\Gamma_3; [\Gamma_1,\Gamma_{13}] = -\Gamma_{11}; [\Gamma_1,\Gamma_{14}] = i\Gamma_{11}; [\Gamma_2,\Gamma_4] = i\Gamma_{10}; [\Gamma_2,\Gamma_5] = \Gamma_8; [\Gamma_2,\Gamma_6] = \Gamma_{10} + i\Gamma_8;\\
&[\Gamma_2,\Gamma_7] = i\Gamma_3;\quad [\Gamma_2,\Gamma_8] = 2\Gamma_2;\quad [\Gamma_2,\Gamma_9] = \Gamma_3;\quad [\Gamma_2,\Gamma_{10}] = 2i\Gamma_2;\quad [\Gamma_2,\Gamma_{13}] = i\Gamma_{12};\quad [\Gamma_2,\Gamma_{14}] = \Gamma_{12}\\
&[\Gamma_3,\Gamma_4] = i\Gamma_7 - \Gamma_{10};\quad [\Gamma_3,\Gamma_5] = i\Gamma_8 + \Gamma_9;\quad [\Gamma_3,\Gamma_6] = i\Gamma_{10} - \Gamma_8 + i\Gamma_9 + \Gamma_7;\quad [\Gamma_3,\Gamma_7] = 2i\Gamma_1 - \Gamma_3;\\
&[\Gamma_3,\Gamma_8] = 2i\Gamma_2 + \Gamma_3;\quad [\Gamma_3,\Gamma_9] = 2\Gamma_1 + i\Gamma_3;\quad [\Gamma_3,\Gamma_{10}] = i\Gamma_3 - 2\Gamma_2;\quad [\Gamma_3,\Gamma_{13}] = i\Gamma_{11} - \Gamma_{12};\\
&[\Gamma_4,\Gamma_7] = 2\Gamma_4;\quad [\Gamma_4,\Gamma_8] = -i\Gamma_6;\quad [\Gamma_4,\Gamma_9] = \Gamma_6;\quad [\Gamma_4,\Gamma_{10}] = -2i\Gamma_4;\quad [\Gamma_4,\Gamma_{11}] = \Gamma_{13};\quad [\Gamma_4,\Gamma_{12}] = -i\Gamma_{13};\\
&[\Gamma_5,\Gamma_7] = -i\Gamma_6; [\Gamma_5,\Gamma_8] = -2\Gamma_5; [\Gamma_5,\Gamma_9] = -2i\Gamma_5; [\Gamma_5,\Gamma_{10}] = -\Gamma_6; [\Gamma_5,\Gamma_{11}] = -i\Gamma_{14}; [\Gamma_5,\Gamma_{12}] = -\Gamma_{14};\\
&[\Gamma_6,\Gamma_7] = -2i\Gamma_4 - \Gamma_6;\quad [\Gamma_6,\Gamma_8] = -\Gamma_6 - 2i\Gamma_5;\quad [\Gamma_6,\Gamma_9] = 2\Gamma_5 - i\Gamma_6;\quad [\Gamma_6,\Gamma_{10}] = -2\Gamma_4 - i\Gamma_6;\\
&[\Gamma_6,\Gamma_{12}] = -i\Gamma_{14} - \Gamma_{13}; [\Gamma_7,\Gamma_8] = -i\Gamma_9 + i\Gamma_{10}; [\Gamma_7,\Gamma_9] = \Gamma_9; [\Gamma_7,\Gamma_{10}] = -i\Gamma_7 - \Gamma_{10}; [\Gamma_7,\Gamma_{11}] = \Gamma_{11};\\
&[\Gamma_7,\Gamma_{12}] = -i\Gamma_{11};\quad [\Gamma_7,\Gamma_{13}] = -\Gamma_{13};\quad [\Gamma_7,\Gamma_{14}] = i\Gamma_{13};\quad [\Gamma_8,\Gamma_9] = -i\Gamma_8 + \Gamma_9;\quad [\Gamma_8,\Gamma_{10}] = \Gamma_9 - i\Gamma_8;\\
&[\Gamma_8,\Gamma_{13}] = i\Gamma_{14};\quad [\Gamma_8,\Gamma_{14}] = \Gamma_{14};\quad [\Gamma_9,\Gamma_{10}] = -\Gamma_8 - \Gamma_7;\quad [\Gamma_9,\Gamma_{11}] = -i\Gamma_{11};\quad [\Gamma_9,\Gamma_{12}] = -\Gamma_{11};\\
&[\Gamma_8,\Gamma_{11}] = -i\Gamma_{12};\quad [\Gamma_8,\Gamma_{12}] = -\Gamma_{12};\quad [\Gamma_9,\Gamma_{13}] = -\Gamma_{14},\quad [\Gamma_9,\Gamma_{14}] = i\Gamma_{14};\quad [\Gamma_{10},\Gamma_{11}] = \Gamma_{12};\\
&[\Gamma_{10},\Gamma_{12}] = -i\Gamma_{12};\quad [\Gamma_{10},\Gamma_{13}] = i\Gamma_{13};\quad [\Gamma_{10},\Gamma_{14}] = \Gamma_{13};\quad [\Gamma_{11},\Gamma_{13}] = -1;\\
&[\Gamma_{11},\Gamma_{14}] = i;\quad [\Gamma_{12},\Gamma_{13}] = i;\quad [\Gamma_{12},\Gamma_{14}] = 1.
\end{aligned} \tag{3.14}$$

Therefore, their use in eq.(3.60) yields the following set of PDEs in λ's:

$$\dot{\lambda}_1 = 4a_1(\lambda_1 - i\lambda_3) + 4(i\lambda_9 - \lambda_7), \tag{3.15}$$

$$\dot{\lambda}_2 = 4a_1(\lambda_2 + i\lambda_3) - 4(\lambda_8 + i\lambda_{10}), \tag{3.16}$$

$$\dot{\lambda}_3 = 2a_1(i\lambda_1 - i\lambda_2 + 2\lambda_3) - 2i\lambda_7 + 2i\lambda_8 - 2\lambda_{10} - 2\lambda_9, \tag{3.17}$$

$$\dot{\lambda}_4 = 4a_1(-\lambda_4 + i\lambda_6) - 4a_0(i\lambda_{10} - \lambda_7), \tag{3.18}$$

$$\dot{\lambda}_5 = -4a_1(\lambda_5 + i\lambda_6) + 4a_0(\lambda_8 + i\lambda_9), \tag{3.19}$$

$$\dot{\lambda}_6 = -2a_1(\lambda_4 - i\lambda_5 + 2i\lambda_6) - 2a_0(\lambda_8 + i\lambda_{10} - i\lambda_9 + i\lambda_7), \tag{3.20}$$

$$\dot{\lambda}_7 = 2a_0(\lambda_1 - i\lambda_3) - 2a_1(i\lambda_{10} - i\lambda_9 + i\lambda_7) - 2\lambda_4 + 2i\lambda_6, \tag{3.21}$$

$$\dot{\lambda}_8 = 2a_0(\lambda_2 - i\lambda_3) - 2\lambda_5 + 2i\lambda_6 - 2ia_1(\lambda_8 + \lambda_{10} - \lambda_9), \tag{3.22}$$

$$\dot{\lambda}_9 = 2a_0(i\lambda_1 + \lambda_3) + 2i\lambda_5 - 2\lambda_6 - 2a_1(i\lambda_7 + \lambda_9 - i\lambda_8), \tag{3.23}$$

$$\dot{\lambda}_{10} = -2a_0(i\lambda_2 + \lambda_3) - 2i\lambda_4 + 2\lambda_6 + 2ia_1(\lambda_7 + \lambda_8 + i\lambda_{10}), \tag{3.24}$$

$$\dot{\lambda}_{11} = 2a_1\lambda_{11} - 2\lambda_{13} + 2\lambda_{14}, \tag{3.25}$$

$$\dot{\lambda}_{12} = 2ia_1\lambda_{11} - 2i\lambda_{13} + 2i\lambda_{14}, \tag{3.26}$$

$$\dot{\lambda}_{13} = 2a_0(-i\lambda_{12} + \lambda_{11}) - 2a_1(\lambda_{13} - i\lambda_{14}), \tag{3.27}$$

$$\dot{\lambda}_{14} = 2a_0(\lambda_{12} + i\lambda_{11}) + a_1(\lambda_{14} - i\lambda_{13}), \tag{3.28}$$

In fact, to solve these fourteen coupled PDEs for complex λ's is very difficult. Thus, here we make certain choices for λ's which facilitate to find solutions of above equations.

From eqs.(3.15), (3.16) and (3.17), we get

$$\dot{\lambda}_3 = i\dot{\lambda}_1 - i\dot{\lambda}_2.$$

and if we $\lambda_3 = c_3$ (a constant, say) consider,then $\dot{\lambda}_3 = 0$, which immediately gives

$$\lambda_1 = \eta(t) + c_1, \quad \lambda_2 = \eta(t) + c_2; \qquad \lambda_3 = c_3, \tag{3.29}$$

where c_1, c_2, and c_3 are complex integration constants. Again from eqs.(3.18), (3.19) and (3.20), we obtain

$$2\dot{\lambda}_6 = \dot{\lambda}_4 - \dot{\lambda}_5. \tag{3.30}$$

and if we $\lambda_6 = c_6$ (a constant, say) consider, then $\dot{\lambda}_6 = 0$, which immediately gives

$$\lambda_4 = \xi(t) + c_4; \qquad \lambda_5 = \xi(t) + c_5; \qquad \lambda_6 = c_6. \tag{3.31}$$

where c_4 and c_5 , c_6 are complex integration constants. Now, in order to find solutions for λ_7, λ_8, λ_9 and λ_{10} subtract eqs.(3.21) and (3.22), we find

$$\dot{\lambda}_7 - \dot{\lambda}_8 = 2a_0(\lambda_1 - \lambda_2) - 2ia_0(-\lambda_7 + \lambda_8) - 2(\lambda_4 + \lambda_5). \tag{3.32}$$

and then using solutions for λ_1, λ_2, λ_4 and λ_5 in eq.(3.30) and again if we set $\lambda_7 = \lambda_8 = \phi(t)$; we get

$$\dot{\lambda}_7 - \dot{\lambda}_8 = 0.$$

which gives

$$\lambda_7 = \phi(t) + c_7; \qquad \lambda_8 = \phi(t) + c_8. \tag{3.33}$$

Here $\phi(t)$ is another arbitrary complex function of time and c_7 and c_8 are complex constants. In the same spirit, adding eq.(3.23) and eq.(3.24),

$$\dot{\lambda}_9 + \dot{\lambda}_{10} = 2ia_0(\lambda_1 - \lambda_2) + 2i(\lambda_5 - \lambda_4) - 2a_1(\lambda_9 + \lambda_{10}). \tag{3.34}$$

and then with the help of eqs.(3.29) or (3.31), we get

$$\lambda_9 = \psi(t) + c_9; \qquad \lambda_{10} = -\psi(t) + c_{10}.$$

where, $\phi(t)$ is one more arbitrary function of time and c_9 and c_{10} are complex constants.

Now for finding the solutions of λ_{11}, λ_{12}, λ_{13} and λ_{14}, we observe from eq.(3.25) to eq.(3.28), that

$$\dot{\lambda}_{11} = i\dot{\lambda}_{12}; \qquad \dot{\lambda}_{13} = i\dot{\lambda}_{14}.$$

which gives

$$\lambda_{11} = \varphi(t) + c_{11}; \qquad \lambda_{12} = -i\varphi(t) + c_{12}; \qquad \lambda_{13} = \chi(t) + c_{13}; \qquad \lambda_{14} = -i\chi(t) + c_{14}). \tag{3.35}$$

We have solved eqs.[(3.15) to (3.28)] in terms of arbitrary functions $\eta, \xi, \phi, \psi, \varphi$ and χ and complex constants, c_i's, $(i = 1, ..., 14)$. If one put back these solutions for λ_i, $(i = 1, ..., 14)$ in eqs.[(3.15) to (3.28)], we obtain a number of constraint relations among c_i's, and η, ξ, ϕ, ψ and φ, which limit the choices of these arbitrary complex quantities. These relations are given as $c_{11} = c_{12}$, $c_{13} = ic_{14}$ and the equations determining arbitrary functions η, ξ, ϕ, ψ and φ are written as

$$\ddot{\eta} - 4(a_1\dot{\eta} + i\dot{\psi} - \dot{\phi}) = 0; \qquad \ddot{\xi} - 4[-a_1\dot{\xi} + a_0(\dot{\phi} - i\dot{\psi})] = 0; \qquad \ddot{\xi} - 2\dot{\xi} = 0,$$
$$\ddot{\phi} - 2(a_0\dot{\eta} - \dot{\xi} - ia_1\dot{\phi}) = 0; \qquad \ddot{\psi} - 2i(a_0\dot{\eta} + \dot{\xi} + ia_1\dot{\psi}) = 0; \qquad \ddot{\varphi} - 2\dot{\varphi} = 0. \tag{3.36}$$

Therefore, after substituting the solutions of λ_i's in the eq.(3.58), the complex integral for a two dimensional complex oscillator becomes

$$\begin{aligned} I = & \frac{1}{2}\eta(p_1^2 + x_2^2) + \frac{1}{2}(c_1p_1^2 + c_2x_2^2) + \frac{1}{2}\xi(x_1^2 + p_2^2) + \frac{1}{2}(c_4x_1^2 + c_5p_2^2) + (c_3p_1x_2 + c_6x_1p_2) \\ & + \phi(x_1p_2 + x_2p_2) + (c_7x_1p_2 + c_8x_2p_2) + \psi(p_1p_2 - x_1x_2) + (c_9p_1p_2 + c_{10}x_2x_1) \\ & + \varphi(p_1 - ix_2) + (c_{11}p_1 + c_{12}x_2) + \chi(x_1 - ip_2) + (c_{13}x_1 + c_{14}p_2). \end{aligned} \tag{3.37}$$

which conforms to condition eq.(3.59) in view of the Poisson bracket eq.(3.7), A discreet and prudent effort are made to obtain exact complex second constant of motion of momentum dependent coupled harmonic oscillator in two-dimensions on an extended complex phase space.

2. Second Example

Consider the case of a shifted harmonic oscillator in x-plane, for which the Hamiltonian is written as [12]

$$H = \frac{1}{2}p^2 + \frac{1}{2}k_0(t)x^2 + k_1(t)x, \tag{3.38}$$

Where k_0 and k_1 are complex function of t. Using the complexification eq.(2.5), the above Hamiltonian can be expressed as

$$\begin{aligned} H &= \frac{1}{2}p_1^2 - \frac{1}{2}x_2^2 + ip_1x_2 + ik_0x_1p_2 + \frac{1}{2}k_0(t)x_1^2 - \frac{1}{2}k_0(t)p_2^2 + k_1(t)x_1 + ik_1tp_2 && (3.39) \\ &= \sum_{m=1}^{8} h_m(t)\Gamma_m(x_1, x_2, p_1, p_2), && (3.40) \end{aligned}$$

and the various Γ's and h(t)'s for the above complex H are expressed as

$$\Gamma_1 = \frac{1}{2}p_1^2;\ \ \Gamma_2 = \frac{1}{2}x_2^2;\ \ \Gamma_3 = p_1x_2;\ \ \Gamma_4 = x_1p_2;\ \ \Gamma_5 = \frac{1}{2}x_1^2;\ \ \Gamma_6 = \frac{1}{2}p_2^2;\ \ \Gamma_7 = x_1;\ \ \Gamma_8 = p_2;$$
$$h_1 = -h_2 = 1;\quad h_3 = i;\quad h_4 = ih_5 = k_0;\quad h_6 = -k_0;\quad h_7 = ih_8 = k_1. \tag{3.41}$$

The dynamical algebra in this case is not closed unless one adds six more phase space functions Γ_l's. The additional Γ_l's and their corresponding h_l's, are given as

$$\Gamma_9 = p_1p_2;\quad \Gamma_{10} = p_1x_1;\quad \Gamma_{11} = p_1;\quad \Gamma_{12} = x_2;\quad \Gamma_{13} = x_1x_2;\quad \Gamma_{14} = p_2x_2$$
$$h_9 = h_{10} = h_{11} = h_{12} = h_{13} = h_{14} = 0. \tag{3.42}$$

Now in the light of modified definition of Poisson bracket for complex systems eq.(3.7), we get large number of (57 no of) nonvanishing Poisson brackets. Therefore using this 57 number of nonvanishing PBs in eq.(3.5) we obtained a set of 14 partial differential equations. As such solution of these 14 coupled partial differential equations for complex λs is solved systematically by some ansatze for λs as done in pervious example. Solutions of λs so obtained are as

$$\lambda_1 = \chi(t) + c_1;\quad \lambda_2 = \chi(t) + c_2;\quad \lambda_3 = c_3,$$
$$\lambda_4 = c_4;\quad \lambda_5 = \phi(t) + c_5;\quad \lambda_6 = \phi(t) + c_6,$$
$$\lambda_7 = \beta(t) + c_7;\quad \lambda_8 = -i\beta(t) + c_8,$$
$$\lambda_9 = -\frac{i}{8}(\dot{\xi} - 8\sigma) + c_9;\quad \lambda_{10} = -\frac{1}{8}(\dot{\xi} + 8\sigma) + c_{10},$$
$$\lambda_{13} = \frac{i}{8}(\dot{\xi} - 8\sigma) + c_{13});\quad \lambda_{14} = -\frac{1}{8}(\dot{\xi} + 8\sigma) + c_{14},$$
$$\lambda_{11} = \alpha(t) + c_{11};\quad \lambda_{12} = -i\alpha(t) + c_{12} \tag{3.43}$$

where $\sigma = \int(2\phi(t) + c_5 + c_6)dt$ and $\alpha(t) = \int(k_1(2\xi(t) + c_1 + c_2))dt$, $\phi(t), \chi(t), \beta(t)$ are arbitrary complex function of time.

Thus, we have solved eqs.(3.43) in terms of arbitrary functions $\chi(t), \phi(t), \sigma(t), \alpha(t)$ and $\beta(t)$ and complex constants, c_i's, $(i = 1, ..., 14)$. Use of these results for λ_i, $(i = 1, ..., 14)$ in eqs. (3.43), we obtain a number of constraint relations among c_i's, and $\chi, \phi, \sigma, \alpha, \beta$, which limit the choices of these arbitrary complex quantities. These relations are given as

$$ic_{13} = -c_{14};\quad ic_9 = -c_{10};\quad ic_8 = c_7,$$
$$ic_{12} = c_{11};\quad c_1 - c_2 = 2ic_3;\quad c_5 - c_6 = 2ic_4, \tag{3.44}$$

and the equations determining arbitrary functions ξ, σ, α and β are written as

$$\ddot{\xi} + 8k_0(2\xi + c_1 - ic_3) = 0;\ \ddot{\sigma} + 8k_0(2\sigma + ic_{13} - c_{10}) = 0,\ \dot{\alpha} - 2k_1(2\xi + c_1 - ic_3) = 0;\ \dot{\beta} + 2k_1(2\sigma - c_{10} + ic_{13}) = 0 \tag{3.45}$$

If we set all c_i's equal to zero, then the solutions (for $k_0 = k_1 = 1$) to these equations can be written as

$$\alpha(t) = \beta(t) = \sigma(t) = \xi(t) = e^{4t}.$$

Therefore, after substituting the solutions of λ_i's in the eq.(3.58), the final form of the integral for a complex shifted harmonic oscillator becomes

$$I = \frac{1}{2}\chi(p_1^2 + x_2^2) + \frac{1}{2}(c_1p_1^2 + c_2x_2^2) + c_3p_1x_2 + c_4x_1p_2 + \frac{1}{2}\phi(x_1^2 + p_2^2) + \frac{1}{2}(c_5x_1^2 + c_6p_2^2)$$
$$+\beta(x_1 - ip_2) + (c_7x_1 + c_8p_2) + (c_9p_1p_2 + c_{13}x_1x_2) + \frac{i}{8}(\dot{\xi} - 8\sigma)(x_1x_2 - p_1p_2)$$
$$-\frac{1}{8}(\dot{\xi} + 8\sigma)(x_1p_1 + x_2p_2) + c_{10}p_1x_1 + c_{14}p_2x_2 + \alpha(p_1 - ix_2) + c_{11}p_1 + c_{12}x_2. \quad (3.46)$$

which conforms to condition eq.(3.59) in view of the Poisson bracket.

3. Third example

Consider the case of a simple harmonic oscillator, for which the Hamiltonian is written as

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2(t)x^2. \quad (3.47)$$

is not constant of motion and what has been found [1] is that this system admits the invariant given by

$$c = [\kappa^2(\frac{x}{\xi})^2 + (\dot{x}\xi - x\dot{\xi})^2].$$

Where auxiliary variable satisfies Milne Pinney's equation, [10] namely

$$\ddot{\xi} + \omega(t)^2\xi = \frac{\kappa^2}{\xi^3}.$$

We search for the complex invariant using the complexification method as discussed in eq.(3.2). The complex version of eqn.(3.47), are obtained by using (3.2) in (3.47), the above Hamiltonian can be expressed as

$$H = \frac{1}{2}p_1^2 - \frac{1}{2}x_2^2 + ip_1x_2 + \omega^2(t)[\frac{1}{2}x_1^2 + ip_2x_1 - \frac{1}{2}p_2^2] \quad (3.48)$$
$$= \sum_{m=1}^{6} h_m(t)\Gamma_m(x_1, x_2, p_1, p_2, t), \quad (3.49)$$

and the various Γ's and $h(t)'s$ for the above complex H are given as

$$\Gamma_1 = \frac{1}{2}p_1^2; \quad \Gamma_2 = \frac{1}{2}x_2^2; \quad \Gamma_3 = p_1x_2; \quad \Gamma_4 = x_1p_2; \quad \Gamma_5 = \frac{1}{2}x_1^2; \quad \Gamma_6 = \frac{1}{2}p_2^2,$$

with

$$h_1 = 1; \quad h_2 = -1; \quad h_3 = i; \quad h_4 = \omega^2(t); \quad h_5 = i\omega^2(t); \quad h_6 = -\omega^2(t). \quad (3.50)$$

The dynamical algebra in this case is not closed. To establish the closure property for the above system, we have to add four more phase space functions Γ_l's. The additional Γ_l's are as follow

$$\Gamma_7 = p_1p_2; \quad \Gamma_8 = p_1x_1; \quad \Gamma_9 = x_1x_2; \quad \Gamma_{10} = p_2x_2;$$
$$h_{11} = h_{12} = h_{13} = h_{14} = 0. \quad (3.51)$$

with corresponding $h_l(t) = 0$. Now in the light of Poisson bracket for complex systems, eq.(3.7), we get large number of (39 no of) nonvanishing Poisson brackets, Therefore using this 39 number of nonvanishing PBs in eq.(3.5) we obtained a set of 10 partial differential equations. As such solution of these 10 coupled partial differential equations for complex $\lambda's$ is solved systematically by some ansatze for $\lambda's$ as done in pervious example. Solutions of $\lambda's$ so obtained are as

$$\begin{aligned}
&\lambda_1 = \rho(t) + c_1; \quad \lambda_2 = \rho(t) + c_2, \quad \lambda_3 = c_3, \quad \lambda_3 = c_3, \\
&\lambda_5 = \eta(t) + c_5, \quad \lambda_6 = \eta(t) + c_6 \quad \lambda_7 = -\frac{i}{8}(\dot{\rho} - 8\sigma) + c_7, \\
&\lambda_8 = -\frac{1}{8}(\dot{\rho} + 8\sigma) + c_8, \quad \lambda_9 = \frac{i}{8}(\dot{\rho} - 8\sigma) + c_9, \quad \lambda_{10} = -\frac{1}{8}(\dot{\rho} + 8\sigma) + c_{10},
\end{aligned} \tag{3.52}$$

Equations from (3.52) have been solved terms of arbitrary functions ρ and σ and complex constants, c_i's, $(i = 1, ..., 10)$. If one put back these solutions for λ_i, $(i = 1, ..., 10)$ in eqs.(3.52), we obtain a number of constraint relations among c_i's, and ρ and σ, which limit the choices of these arbitrary complex quantities. These relations are given as $ic_9 = -c_{10}, \quad c_8 = ic_7$ other equations are $ic_3 = c_1 - c_2, \quad c_5 - c_6 = 2ic_4$ and the equations determining arbitrary functions $\rho(t)$ and $\sigma(t)$ and φ are

$$\begin{aligned}
&\ddot{\rho} + 16\omega^2\rho + 8\omega^2(c_1 + c_2) = 0, \\
&\ddot{\sigma} + 16\omega^2\sigma + 8\omega^2(ic_9 - c_8) = 0,
\end{aligned} \tag{3.53}$$

If we set all c_i's equal to zero, then the solutions (for $\omega = 1$) to these equations can be written as

$$\rho(t) = \sigma(t) = e^{4t}.$$

Therefore, after substituting the solutions of λ_i's in the eq.(3.58), the complex integral for a two dimensional complex oscillator becomes

$$\begin{aligned}
I = &\frac{1}{2}\rho(p_1^2 + x_2^2) + \frac{1}{2}(c_1 p_1^2 + c_2 x_2^2) + \frac{1}{2}\eta(x_1^2 + p_2^2) + \frac{1}{2}(c_5 x_1^2 + c_6 p_2^2) + (c_3 p_1 x_2 + c_4 x_1 p_2) - \frac{i}{8}(\dot{\rho} - 8\sigma) \\
&(p_1 p_2 - x_1 x_2) - \frac{1}{8}(\dot{\rho} + 8\sigma)(x_1 p_1 + x_2 p_2) + (c_7 p_1 p_2 + c_9 x_2 x_1) + (c_8 p_1 x_1 + c_{10} p_2 x_2).
\end{aligned} \tag{3.54}$$

which conforms to condition eq.(3.59) in view of the Poisson bracket eq.(3.7), In this work a diffident attempt has been made to find the expression exact complex second constant of motion of harmonic oscillator in one-dimensions on an extended complex phase space.

3.2 Construction of complex invariants in two dimensions

Physical dynamical systems in higher dimensions are always interesting. On the classical level only a few studies of two-dimensional real phase systems have been reported in recent past [13]. Here, in the present work we carry out the extended phase plane approach to obtain exact complex invariants of a two-dimensional classical dynamical system [14, 15]. While making a transition from one-dimensional to two- or three-dimensional dynamical systems not only the number of coordinates are increased but also inter-dimensional and position-momentum coupling terms also important to specify the dynamics of the systems.

The method for two-dimensions

As we have already discussed it in section 3.2.2 of chapter 3, that for a two dimensional real phase space (x, y, p_x, p_y, t), which may be transformed into a complex space $(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t)$, by defining position and momenta variables as

$$x = x_1 + ip_3; \quad y = x_2 + ip_4; \quad p_x = p_1 + ix_3; \quad p_y = p_2 + ix_4; \tag{3.55}$$

The presence of variables (x_3, x_4, p_3, p_4) in the above transformation eq. can be regarded as some sort of coordinate-momentum interaction of the dynamical system. In the Lie-algebraic approach, the complex Hamiltonian in two dimension $H(p_x, p_y, x, y, t)$ of the system can expressed as

$$H = \sum_n h_n(t)\Gamma_n(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t), \tag{3.56}$$

where the set of functions $\{\Gamma_1, ..., \Gamma_n\}$ are not explicitly time dependent and $h_n(t)$ are complex coefficient functions of time. The Γ_n's in eq.(3.56) generate a closed dynamical algebra, implies $[\Gamma_n, \Gamma_m] = \sum_l C^l_{nm}\Gamma_l$, where C^l_{nm} are the complex structure constants of the algebra. However for two dimensional real systems Poison bracket turns out to be

$$[f, g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p_x} - \frac{\partial f}{\partial p_x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial p_y} - \frac{\partial f}{\partial p_y}\frac{\partial g}{\partial y} \tag{3.57}$$

If the Γ_n's in eq.(3.56) are not sufficient to close the algebra then the set of Γ_n must be extended by adding new Γ_l's, such that $\Gamma_l = [\Gamma_n, \Gamma_m]$, until the closure is obtained along with additional $h_l(t)$'s which are taken to be zero. Since the complex dynamical integral I is also a part of Lie algebra, then one can express this as

$$I(t) = \sum_k \lambda_k(t)\Gamma_k(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t), \tag{3.58}$$

where $\lambda_k(t)$'s are time dependent complex coefficients. Thus by using eq.(3.56) and eq.(3.58) for H and I respectively in the invariance condition

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H] = 0, \tag{3.59}$$

we get a system of linear, first order differential equations, namely

$$\dot{\lambda}_r + \sum_n \left[\sum_m C^r_{nm} h_m(t)\right]\lambda_n = 0, \tag{3.60}$$

in λ_n's. Therefore, the solutions of these differential equations in turn provide classical complex integrals of a given system from eq.(3.58). In the next following sections we will use the Lie algebraic methods to obtain complex invariants of different dimensional classical complex Hamiltonian systems.

The expression for two-dimensional Hamiltonian $H(x, y, p_x, p_y, t)$ system in complex space can be written by using eq.(3.55), as

$$H = H_1(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t) + iH_2(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t). \tag{3.61}$$

From eq.(3.55) one can easily obtain

$$\begin{aligned} \frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_3}; \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4}; \\ \frac{\partial}{\partial p_x} = \frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3}; \qquad \frac{\partial}{\partial p_y} = \frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}. \end{aligned} \tag{3.62}$$

The equations of motion for complex Hamilton's are H, eq.(3.61), can be written as

$$\begin{aligned} \dot{x}_1 = \left(\frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_3}\right); \quad \dot{p}_3 = \left(\frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_3}\right); \quad \dot{x}_2 = \left(\frac{\partial H_1}{\partial p_2} + \frac{\partial H_2}{\partial x_4}\right); \quad \dot{p}_4 = \left(\frac{\partial H_2}{\partial p_2} - \frac{\partial H_1}{\partial x_4}\right); \\ \dot{p}_1 = -\left(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_3}\right); \dot{x}_3 = -\left(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_3}\right); \dot{p}_2 = -\left(\frac{\partial H_1}{\partial x_2} + \frac{\partial H_2}{\partial p_4}\right); \dot{x}_4 = -\left(\frac{\partial H_2}{\partial x_2} - \frac{\partial H_1}{\partial p_4}\right). \end{aligned} \tag{3.63}$$

If the H, eq.(3.61), is an analytic function of complex variables, then H_1 and H_2 satisfy the Cauchy-Riemann conditions [13] and after invoking such analyticity conditions, eq.(3.63) reduces

$$\begin{aligned} \dot{x}_1 = 2\frac{\partial H_1}{\partial p_1}; \; \dot{p}_1 = -2\frac{\partial H_1}{\partial x_1}; \; \dot{x}_2 = 2\frac{\partial H_1}{\partial p_2}; \; \dot{p}_2 = -2\frac{\partial H_1}{\partial x_2}; \\ \dot{x}_3 = 2\frac{\partial H_1}{\partial p_3}; \; \dot{p}_3 = -2\frac{\partial H_1}{\partial x_3}; \; \dot{x}_4 = 2\frac{\partial H_1}{\partial p_4}; \; \dot{p}_4 = -2\frac{\partial H_1}{\partial x_4}. \end{aligned} \tag{3.64}$$

Note that $(x_1, p_1), (x_2, p_2), (x_3, p_3)$ and (x_4, p_4) constitute canonical pairs. Now consider a complex phase space function $I(x, y, p_x, p_y, t)$ as

$$I = I_1(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t) + iI_2(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t). \tag{3.65}$$

Thus for function I to be the TD (time dependent) dynamical invariant of the system in complex phase space, then this must conform the following invariance condition:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H]_{PB} = 0, \tag{3.66}$$

where $[.,.]$ is the PB, which in view of the definition, eq.(3.55), turns out to be

$$\begin{aligned} [A, B]_{(x,p)} &= [A, B]_{(x_1,p_1)} - i[A, B]_{(x_1,x_3)} - i[A, B]_{(p_3,p_1)} - [A, B]_{(p_3,x_3)} \\ &\quad + [A, B]_{(x_2,p_2)} - i[A, B]_{(x_2,x_4)} - i[A, B]_{(p_4,p_2)} - [A, B]_{(p_4,x_4)}. \end{aligned} \tag{3.67}$$

this means computation of Poisson bracket in case of complex Hamiltonian systems becomes a bit tedious. The method Lie-algebraic approach, has been same and it is briefly described in previous section for one dimensional complex invariants of classical dynamical systems. In the next subsection we shall make use of the procedure given above to obtain complex invariant of a classical complex Hamiltonian system.

3.2.1 Some examples from oscillator system

1. Simple harmonic oscillator system

Consider a simple harmonic oscillator in two dimensions [16], whose Hamiltonian is given by

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\omega^2(t)(x^2 + y^2). \tag{3.68}$$

Using eq.(3.55), the above Hamiltonian can be expressed as

$$\begin{aligned} H = & \frac{1}{2}p_1^2 - \frac{1}{2}x_3^2 + \frac{1}{2}p_2^2 - \frac{1}{2}x_4^2 + ip_2x_4 + ip_1x_3 + i\omega^2(t)p_3x_1 \\ & + i\omega^2(t)p_4x_2 + \frac{1}{2}\omega^2(t)x_1^2 - \frac{1}{2}\omega^2(t)p_3^2 + \frac{1}{2}\omega^2(t)x_2^2 - \frac{1}{2}\omega^2(t)p_4^2 \\ = & \sum_{m=1}^{12} h_m(t)\Gamma_m(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4), \end{aligned} \tag{3.69}$$

and the various Γ's and h(t)'s for the above complex H are given as

$$\begin{aligned} &\Gamma_1 = \frac{1}{2}p_1^2; \quad \Gamma_2 = \frac{1}{2}x_3^2; \quad \Gamma_3 = \frac{1}{2}p_2^2; \quad \Gamma_4 = \frac{1}{2}x_4^2, \quad \Gamma_5 = p_1x_3; \quad \Gamma_6 = p_2x_4; \\ &\Gamma_7 = p_3x_1; \quad \Gamma_8 = p_4x_2; \quad \Gamma_9 = \frac{1}{2}x_1^2; \quad \Gamma_{10} = \frac{1}{2}p_3^2; \quad \Gamma_{11} = \frac{1}{2}x_2^2; \quad \Gamma_{12} = \frac{1}{2}p_4^2, \\ &h_1 = h_3 = 1 = -h_2 = -h_4 = 1; \; h_5 = h_6 = i; \; ih_7 = ih_8 = h_9 = h_{11} = \omega^2(t) = -h_{10} = -h_{12}. \end{aligned} \tag{3.70}$$

The dynamical algebra in this case is not closed. To find closure property for the above system, we add eight more phase space functions (Γ_l)'s. The additional (Γ_l)'s are as follow

$$\begin{aligned} &\Gamma_{13} = p_1p_3; \quad \Gamma_{14} = p_1x_1; \quad \Gamma_{15} = x_1x_3; \quad \Gamma_{16} = p_3x_3, \\ &\Gamma_{17} = p_2p_4; \quad \Gamma_{18} = p_2x_2; \quad \Gamma_{19} = p_4x_4; \quad \Gamma_{20} = x_2x_4. \end{aligned} \tag{3.71}$$

with corresponding $h_l(t) = 0$. Now in the light of Poisson bracket for complex systems, eq.(3.67), we get large number of nonvanishing Poisson brackets, namely

$$\begin{aligned}
&[\Gamma_1, \Gamma_7] = -\Gamma_{13} + i\Gamma_{14}; \quad [\Gamma_1, \Gamma_9] = -\Gamma_{14}; \quad [\Gamma_1, \Gamma_{10}] = i\Gamma_{13}; \quad [\Gamma_1, \Gamma_{13}] = 2i\Gamma_1, \\
&[\Gamma_1, \Gamma_{14}] = -2\Gamma_1; \quad [\Gamma_1, \Gamma_{15}] = -\Gamma_5; \quad [\Gamma_1, \Gamma_{16}] = i\Gamma_5; \quad [\Gamma_2, \Gamma_7] = \Gamma_{15} + i\Gamma_{16}, \\
&[\Gamma_2, \Gamma_9] = i\Gamma_{15}; \quad [\Gamma_2, \Gamma_{10}] = \Gamma_{16}; \quad [\Gamma_2, \Gamma_{13}] = \Gamma_5; \quad [\Gamma_2, \Gamma_{14}] = i\Gamma_5, \\
&[\Gamma_2, \Gamma_{15}] = 2i\Gamma_2; \quad [\Gamma_2, \Gamma_{16}] = 2\Gamma_2; \quad [\Gamma_3, \Gamma_8] = -\Gamma_{17} + \imath\Gamma_{16}; \quad [\Gamma_3, \Gamma_{11}] = -\Gamma_{18}, \\
&[\Gamma_3, \Gamma_{12}] = i\Gamma_{17}; \quad [\Gamma_3, \Gamma_{17}] = 2i\Gamma_3; \quad [\Gamma_3, \Gamma_{18}] = -2\Gamma_3; \quad [\Gamma_3, \Gamma_{19}] = i\Gamma_6, \\
&[\Gamma_3, \Gamma_{20}] = -\Gamma_6; \quad [\Gamma_4, \Gamma_8] = i\Gamma_{19} + \Gamma_{20}; \quad [\Gamma_4, \Gamma_{11}] = i\Gamma_{20}; \quad [\Gamma_4, \Gamma_{12}] = \Gamma_{19}; \\
&[\Gamma_4, \Gamma_{17}] = \Gamma_6; \quad [\Gamma_4, \Gamma_{18}] = i\Gamma_6; \quad [\Gamma_4, \Gamma_{19}] = 2\Gamma_4; \quad [\Gamma_4, \Gamma_{20}] = 2i\Gamma_4, \\
&[\Gamma_5, \Gamma_7] = i\Gamma_{13} + \Gamma_{14} + i\Gamma_{15} - \Gamma_{16}; \quad [\Gamma_5, \Gamma_9] = i\Gamma_{14} - \Gamma_{15}; \quad [\Gamma_5, \Gamma_{10}] = \Gamma_{13} + i\Gamma_{16}, \\
&[\Gamma_5, \Gamma_{13}] = 2\Gamma_1 + i\Gamma_1; \quad [\Gamma_5, \Gamma_{14}] = 2i\Gamma_1 - \Gamma_5; \quad [\Gamma_5, \Gamma_{15}] = -2\Gamma_{14} + i\Gamma_5, \\
&[\Gamma_5, \Gamma_{16}] = 2i\Gamma_2 + \Gamma_5; [\Gamma_6, \Gamma_8] = i\Gamma_{17} + \Gamma_{18} - \Gamma_{19} + i\Gamma_{20}; \quad [\Gamma_6, \Gamma_{11}] = i\Gamma_{18} - \Gamma_{20}, \\
&[\Gamma_6, \Gamma_{12}] = \Gamma_{17} + i\Gamma_{19}; [\Gamma_6, \Gamma_{17}] = 2\Gamma_3 + i\Gamma_6; \quad [\Gamma_6, \Gamma_{18}] = 2i\Gamma_3 - \Gamma_6, \\
&[\Gamma_6, \Gamma_{19}] = 2i\Gamma_4 + \Gamma_6; \quad [\Gamma_6, \Gamma_{20}] = -2\Gamma_4 + i\Gamma_6; [\Gamma_7, \Gamma_{13}] = -i\Gamma_7 + 2\Gamma_{10}, \\
&[\Gamma_7, \Gamma_{14}] = \Gamma_7 - 2i\Gamma_9; \quad [\Gamma_7, \Gamma_{15}] = -i\Gamma_7 - 2\Gamma_9; [\Gamma_7, \Gamma_{16}] = -\Gamma_7 - 2i\Gamma_{10}, \\
&[\Gamma_8, \Gamma_{17}] = -i\Gamma_8 + 2\Gamma_{12}; \quad [\Gamma_8, \Gamma_{18}] = \Gamma_8 - 2i\Gamma_{11}; [\Gamma_8, \Gamma_{19}] = -\Gamma_8 - 2i\Gamma_{12}, \\
&[\Gamma_8, \Gamma_{20}] = -i\Gamma_8 - 2\Gamma_{11}; \quad [\Gamma_9, \Gamma_{13}] = \Gamma_7; \quad [\Gamma_9, \Gamma_{14}] = 2\Gamma_9; [\Gamma_9, \Gamma_{15}] = -2i\Gamma_9, \\
&[\Gamma_9, \Gamma_{16}] = -i\Gamma_7; \quad [\Gamma_{10}, \Gamma_{13}] = -2i\Gamma_{10}; \quad [\Gamma_{10}, \Gamma_{14}] = -i\Gamma_7; [\Gamma_{10}, \Gamma_{15}] = -\Gamma_7, \\
&[\Gamma_{10}, \Gamma_{16}] = -2\Gamma_{10}; \quad [\Gamma_{11}, \Gamma_{17}] = \Gamma_8; \quad [\Gamma_{11}, \Gamma_{18}] = 2\Gamma_{11}; [\Gamma_{11}, \Gamma_{19}] = -i\Gamma_8, \\
&[\Gamma_{11}, \Gamma_{20}] = -2i\Gamma_{11}; \quad [\Gamma_{12}, \Gamma_{17}] = -2i\Gamma_{12}; \quad [\Gamma_{12}, \Gamma_{18}] = -i\Gamma_8; [\Gamma_{12}, \Gamma_{19}] = -2\Gamma_{12},
\end{aligned}$$

$$
\begin{aligned}
&[\Gamma_{12}, \Gamma_{20}] = -\Gamma_8; \quad [\Gamma_{13}, \Gamma_{14}] = -\Gamma_{13} - i\Gamma_{14}; \quad [\Gamma_{13}, \Gamma_{15}] = -\Gamma_{14} - \Gamma_{16},\\
&[\Gamma_{13}, \Gamma_{16}] = -\Gamma_{13} + i\Gamma_{16}; \quad [\Gamma_{14}, \Gamma_{15}] = -i\Gamma_{14} - \Gamma_{15}; \quad [\Gamma_{14}, \Gamma_{16}] = -i\Gamma_{13} - i\Gamma_{15};\\
&[\Gamma_{15}, \Gamma_{16}] = \Gamma_{15} - i\Gamma_{16}; \quad [\Gamma_{17}, \Gamma_{18}] = -\Gamma_{17} - i\Gamma_{18}; \quad [\Gamma_{17}, \Gamma_{19}] = -\Gamma_{17} + i\Gamma_{19};\\
&[\Gamma_{17}, \Gamma_{20}] = -\Gamma_{18} - \Gamma_{19}; \quad [\Gamma_{18}, \Gamma_{19}] = -i\Gamma_{17} + \imath\Gamma_{20}; \quad [\Gamma_{18}, \Gamma_{20}] = -i\Gamma_{18} - \Gamma_{20};\\
&[\Gamma_{19}, \Gamma_{20}] = i\Gamma_{19} - \Gamma_{20}.
\end{aligned}
\tag{3.72}
$$

Therefore, their use in eq.(3.59) yields the following set of PDEs in λ's:

$$\dot{\lambda}_1 = -4(i\lambda_{13} - \lambda_{14}); \quad \dot{\lambda}_2 = 4(i\lambda_{15} + \lambda_{16}), \tag{3.73}$$

$$\dot{\lambda}_3 = -4(i\lambda_{17} - \lambda_{18}); \quad \dot{\lambda}_4 = 4(\lambda_{19} + i\lambda_{20}), \tag{3.74}$$

$$\dot{\lambda}_5 = 2(\lambda_{13} + i\lambda_{14} + \lambda_{15} - i\lambda_{16}); \quad \dot{\lambda}_6 = 2(\lambda_{17} + i\lambda_{18} - i\lambda_{19} + \lambda_{20}), \tag{3.75}$$

$$\dot{\lambda}_7 = -2\omega^2(\lambda_{13} + i\lambda_{14} + \lambda_{15} - i\lambda_{16}); \quad \dot{\lambda}_8 = -2\omega^2(\lambda_{17} + i\lambda_{18} - i\lambda_{19} + \lambda_{20}), \tag{3.76}$$

$$\dot{\lambda}_9 = -4\omega^2(\lambda_{14} - i\lambda_{15}); \quad \dot{\lambda}_{10} = -4\omega^2(i\lambda_{13} + \lambda_{16}), \tag{3.77}$$

$$\dot{\lambda}_{11} = -4\omega^2(\lambda_{18} - i\lambda_{20}); \quad \dot{\lambda}_{12} = -4\omega^2(i\lambda_{17} + i\lambda_{19}), \tag{3.78}$$

$$\dot{\lambda}_{13} = -2\omega^2(i\lambda_1 + \lambda_5 - \lambda_7 + i\lambda_{10}); \quad \dot{\lambda}_{14} = -2\omega^2(\lambda_1 - i\lambda_5 + i\lambda_7 - \lambda_9), \tag{3.79}$$

$$\dot{\lambda}_{15} = 2\omega^2(i\lambda_2 - \lambda_5 + \lambda_7 + i\lambda_9); \quad \dot{\lambda}_{16} = -2\omega^2(\lambda_2 + i\lambda_5 - i\lambda_7 - \lambda_{10}), \tag{3.80}$$

$$\dot{\lambda}_{17} = -2\omega^2(i\lambda_3 + \lambda_6 - \lambda_8 + i\lambda_{12}); \quad \dot{\lambda}_{18} = -2\omega^2(\lambda_3 - i\lambda_6 + i\lambda_8 - \lambda_{11}), \tag{3.81}$$

$$\dot{\lambda}_{19} = -2\omega^2(\lambda_4 + i\lambda_6 - i\lambda_8 - \lambda_{12}); \quad \dot{\lambda}_{20} = 2\omega^2(i\lambda_4 - \lambda_6 + \lambda_8 + i\lambda_{11}). \tag{3.82}$$

In fact, to solve these 20 coupled PDEs for complex λ's is very difficult. Thus, here we make certain choices for λ's which facilitate to find solutions of above equations.
From above eqs. we get, $\omega^2\dot{\lambda}_5 + \dot{\lambda}_7 = 0$; and consider $\dot{\lambda}_5 = 0$;, which immediately gives

$$\lambda_5 = c_5; \quad \lambda_7 = c_7. \tag{3.83}$$

where c_5 and c_7 are complex integration constants. Similarly, we obtain

$$\lambda_6 = c_6; \quad \lambda_8 = c_8. \tag{3.84}$$

Thus eqs.(3.75) it become

$$i\lambda_{13} - \lambda_{14} + i\lambda_{15} + \lambda_{16} = 0,$$

and the above equation, after using eqs. (3.73), reduces to $\dot{\lambda}_1 = \dot{\lambda}_2$; which on integration gives

$$\lambda_1 = \rho(t) + c_1; \quad \lambda_2 = \rho(t) + c_2, \tag{3.85}$$

where $\rho(t)$ is some arbitrary complex function of time and c_1 and c_2 are arbitrary complex constants. Similarly, from eqs.(3.74), after using eqs.(3.73), we find

$$\lambda_3 = \xi(t) + c_3; \quad \lambda_4 = \xi(t) + c_4. \tag{3.86}$$

Again, $\xi(t)$ is some arbitrary complex function of time and c_3 and c_4 are arbitrary complex constants. Now, in order to find solutions for λ_9 and λ_{10}, we can find from above eqn. $\dot{\lambda}_9 = \dot{\lambda}_{10}$, which gives

$$\lambda_9 = \eta(t) + c_9; \quad \lambda_{10} = \eta(t) + c_{10}. \tag{3.87}$$

Here $\eta(t)$ is another arbitrary complex function of time and c_9 and c_{10} are complex constants. In the same spirit, we get

$$\lambda_{11} = \phi(t) + c_{11}; \quad \lambda_{12} = \phi(t) + c_{12}. \tag{3.88}$$

where, $\phi(t)$ is one more arbitrary function of time and c_{11} and c_{12} are complex constants.
Now for finding the solutions of λ_{13} and λ_{14}, subtract i times eq.(3.79) from eq.(3.79) and after using eq.(3.87), we get

$$\dot{\lambda}_{13} - i\dot{\lambda}_{14} = -2i(\lambda_9 + \lambda_{10}).$$

or

$$i\dot{\lambda}_{13} + \dot{\lambda}_{14} = 2(2\eta + c_9 + c_{10}). \tag{3.89}$$

On the other hand, time derivative of eq.(3.73) is written as

$$\ddot{\lambda}_1 = 4(-i\lambda_{13} + \lambda_{14}) = \ddot{\rho}. \tag{3.90}$$

Hence using eqs.(3.89) and (3.90), one immediately get

$$\lambda_{13} = \frac{i}{8}(\dot{\rho} - 8\sigma) + c_{13}; \quad and \quad \lambda_{14} = \frac{1}{8}(\dot{\rho} + 8\sigma) + c_{14}, \tag{3.91}$$

where $\sigma = \int (2\eta(t) + c_9 + c_{10}) dt$.
Similarly, we obtain solutions for $(\lambda_{15}, \lambda_{16})$, $(\lambda_{17}, \lambda_{18})$ and $(\lambda_{19}, \lambda_{20})$ respectively as

$$\lambda_{15} = -\frac{i}{8}(\dot{\rho} - 8\sigma) + c_{15}; \qquad \lambda_{16} = \frac{1}{8}(\dot{\rho} + 8\sigma) + c_{16}, \tag{3.92}$$

$$\lambda_{17} = \frac{i}{8}(\dot{\xi} - 8\theta) + c_{17}; \qquad \lambda_{18} = \frac{1}{8}(\dot{\xi} + 8\theta) + c_{18}, \tag{3.93}$$

$$\lambda_{19} = \frac{1}{8}(\dot{\xi} + 8\theta) + c_{19}; \qquad \lambda_{20} = -\frac{i}{8}(\dot{\xi} - 8\theta) + c_{20}. \tag{3.94}$$

where $\theta = \int(2\xi(t) + c_{11} + c_{12})dt$.

We have solved eqs.(3.73)-(3.82) in terms of arbitrary functions ρ, ξ, η and ϕ and complex constants, c_i's, $(i = 1, ..., 20)$. If one put back these solutions for λ_i, $(i = 1, ..., 20)$ in eqs.(3.73)-(3.82), we obtain a number of constraint relations among c_i's, and ρ, ξ, η, ϕ, which limit the choices of these arbitrary complex quantities. If we set all c_i's equal to zero, then these relations are given with the equations determining arbitrary functions ρ, ξ, η and ϕ are written as

$$\ddot{\rho} + 16\omega^2\rho = 0; \quad \ddot{\xi} + 16\omega^2\xi = 0; \quad \ddot{\eta} + 16\omega^2\eta = 0; \quad \ddot{\phi} + 16\omega^2\phi = 0. \tag{3.95}$$

then the solutions (for $\omega = 1$) to these equations can be written as

$$\rho(t) = \xi(t) = \eta(t) = \phi(t) = e^{4t}.$$

Therefore, after substituting the solutions of λ_i's in the eq.(1.10), the complex invariant for a two dimensional complex oscillator becomes

$$\begin{aligned} I = &\frac{1}{2}\rho(p_1^2 + x_3^2) + \frac{1}{2}\xi(p_2^2 + x_4^2) + \frac{1}{2}\eta(x_1^2 + p_3^2) + \frac{1}{2}\phi(x_2^2 + p_4^2) + \frac{i}{8}(\dot{\rho} - 8\sigma)(p_1p_3 - x_1x_3) \\ &+ \frac{1}{8}(\dot{\rho} + 8\sigma)(x_1p_1 + x_3p_3) + \frac{i}{8}(\dot{\xi} - 8\theta)(p_2p_4 - x_2x_4) + \frac{1}{8}(\dot{\xi} + 8\theta)(p_2x_2 + p_4x_4) \end{aligned} \tag{3.96}$$

which conforms to condition eq.(3.66) in view of the Poisson bracket eq.(3.67).

2. Shifted harmonic oscillator system

Consider a shifted harmonic oscillator systems in two-dimensions [9, 17], whose Hamiltonian is given by

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\kappa_1(t)^2(x^2 + y^2) + \kappa_2(t)xy. \tag{3.97}$$

The complex Hamiltonian on extended phase space can written as can be expressed as

$$\begin{aligned} H = &\frac{1}{2}p_1^2 - \frac{1}{2}x_3^2 + ip_1x_3 + \frac{1}{2}p_2^2 - \frac{1}{2}x_4^2 + ip_2x_4 + \kappa_1(t)^2\Bigg[\frac{1}{2}x_1^2 - \frac{1}{2}p_3^2 + ip_3x_1 + \frac{1}{2}x_2^2 - \frac{1}{2}p_4^2 \\ &+ ix_2p_4\Bigg] + \kappa_2(t)^2[x_1x_2 + ip_3x_2 + ip_4x_1 - p_3p_4] = \sum_{m=1}^{16} h_m(t)\Gamma_m(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4) \end{aligned} \tag{3.98}$$

and the various Γ's and h(t)'s for the above complex H are given as

$$\begin{aligned} &\Gamma_1 = \frac{1}{2}p_1^2;\ \Gamma_2 = \frac{1}{2}x_3^2;\ \Gamma_3 = \frac{1}{2}p_2^2;\ \Gamma_4 = \frac{1}{2}x_4^2;\ \Gamma_5 = p_1x_3;\ \Gamma_6 = p_2x_4;\ \Gamma_7 = p_3x_1;\ \Gamma_8 = p_4x_2, \\ &\Gamma_9 = \frac{1}{2}x_1^2;\ \Gamma_{10} = \frac{1}{2}p_3^2;\ \Gamma_{11} = \frac{1}{2}x_2^2;\ \Gamma_{12} = \frac{1}{2}p_4^2;\ \Gamma_{13} = x_1x_2;\ \Gamma_{14} = p_3p_4;\ \Gamma_{15} = p_3x_2; \Gamma_{16} = p_4x_1; \end{aligned} \tag{3.99}$$

with

$$\begin{aligned} &h_1 = h_3 = 1 = -h_2 = h_4 = 1,\ h_5 = h_6 = i;\ ih_7 = ih_8 = \kappa_1^2 = h_9 = h_{11}, \\ &h_{10} = h_{12} = -\kappa_1^2;\ h_{13} = \kappa_2 = -h_{14} = ih_{15};\ h_{16} = i\kappa_2. \end{aligned} \tag{3.100}$$

The dynamical algebra in this case is not closed. To find closure property for the above system, we add twenty more phase space functions Γ_l's. The additional (Γ_l)'s are as follow

$$\Gamma_{17} = p_1p_3; \quad \Gamma_{18} = p_1x_1; \quad \Gamma_{19} = p_4p_1; \quad \Gamma_{20} = p_3x_3;$$

$$\Gamma_{21}=x_1x_3;\quad \Gamma_{22}=x_3x_2;\quad \Gamma_{23}=x_2p_2;\quad \Gamma_{24}=p_2p_4;$$
$$\Gamma_{25}=p_3p_2;\quad \Gamma_{26}=x_2x_4;\quad \Gamma_{27}=p_4x_4;\quad \Gamma_{28}=x_1x_4;$$
$$\Gamma_{29}=x_3p_4;\quad \Gamma_{30}=x_1p_2;\quad \Gamma_{31}=p_1x_2;\quad \Gamma_{32}=p_3x_4,$$
$$\Gamma_{33}=p_1x_4;\quad \Gamma_{34}=p_1p_2;\quad \Gamma_{35}=x_3x_4;\quad \Gamma_{36}=p_2x_3; \tag{3.101}$$

with corresponding $h_l(t)=0$. Now in the light of Poisson bracket for complex systems, eq.(3.67), we get large number of non-vanishing Poisson brackets, namely

$$[\Gamma_1,\Gamma_7]=-\Gamma_{17}+i\Gamma_{18};\quad [\Gamma_1,\Gamma_9]=-\Gamma_{18};\quad [\Gamma_1,\Gamma_{10}]=i\Gamma_{17};\quad [\Gamma_1,\Gamma_{13}]=-\Gamma_{31};$$
$$[\Gamma_1,\Gamma_{14}]=i\Gamma_{19};\quad [\Gamma_1,\Gamma_{15}]=i\Gamma_{31};\quad [\Gamma_1,\Gamma_{16}]=-\Gamma_{19};\quad [\Gamma_1,\Gamma_{17}]=2i\Gamma_1;$$
$$[\Gamma_1,\Gamma_{18}]=-2\Gamma_1;\quad [\Gamma_1,\Gamma_{20}]=i\Gamma_5;\quad [\Gamma_1,\Gamma_{21}]=-\Gamma_5;\quad [\Gamma_1,\Gamma_{25}]=i\Gamma_{34};$$
$$[\Gamma_1,\Gamma_{28}]=-\Gamma_{33};\quad [\Gamma_1,\Gamma_{30}]=-\Gamma_{34};\quad [\Gamma_1,\Gamma_{32}]=i\Gamma_{33}$$
$$[\Gamma_2,\Gamma_7]=-i\Gamma_{20}+\Gamma_{21};\quad [\Gamma_2,\Gamma_9]=i\Gamma_{21};\quad [\Gamma_2,\Gamma_{10}]=i\Gamma_{20};\quad [\Gamma_2,\Gamma_{13}]=i\Gamma_{22};$$
$$[\Gamma_2,\Gamma_{14}]=\Gamma_{29};\quad [\Gamma_2,\Gamma_{15}]=\Gamma_{22};\quad [\Gamma_2,\Gamma_{16}]=i\Gamma_{29};\quad [\Gamma_2,\Gamma_{17}]=\Gamma_5;$$
$$[\Gamma_2,\Gamma_{18}]=i\Gamma_5;\quad [\Gamma_2,\Gamma_{20}]=2\Gamma_2;\quad [\Gamma_2,\Gamma_{21}]=2i\Gamma_2;\quad [\Gamma_2,\Gamma_{25}]=\Gamma_{36};$$
$$[\Gamma_2,\Gamma_{28}]=i\Gamma_{35};\quad [\Gamma_2,\Gamma_{30}]=i\Gamma_{36};\quad [\Gamma_2\Gamma_{32}]=\Gamma_{35}$$
$$[\Gamma_3,\Gamma_8]=i\Gamma_{23}+\Gamma_{24};\quad [\Gamma_3,\Gamma_{11}]=-\Gamma_{23};\quad [\Gamma_3,\Gamma_{12}]=i\Gamma_{24}\quad [\Gamma_3,\Gamma_{13}]=-\Gamma_{30};$$
$$[\Gamma_3,\Gamma_{14}]=i\Gamma_{25};\quad [\Gamma_3,\Gamma_{15}]=-\Gamma_{25};\quad [\Gamma_3,\Gamma_{16}]=i\Gamma_{30};\quad [\Gamma_3,\Gamma_{19}]=i\Gamma_{34};$$
$$[\Gamma_3,\Gamma_{22}]=-\Gamma_{36};\quad [\Gamma_3,\Gamma_{23}]=-2\Gamma_3;\ [\Gamma_3,\Gamma_{24}]=2i\Gamma_3;$$
$$[\Gamma_3,\Gamma_{26}]=-\Gamma_6;\quad [\Gamma_3,\Gamma_{27}]=i\Gamma_6;\quad [\Gamma_3,\Gamma_{29}]=i\Gamma_{36};\quad [\Gamma_3,\Gamma_{31}]=-\Gamma_{34};$$
$$[\Gamma_4,\Gamma_8]=i\Gamma_{27}+\Gamma_{26};\quad [\Gamma_4,\Gamma_{11}]=i\Gamma_{26};\quad [\Gamma_4,\Gamma_{12}]=\Gamma_{27}\quad [\Gamma_4,\Gamma_{13}]=i\Gamma_{28};$$
$$[\Gamma_4,\Gamma_{14}]=\Gamma_{32};\quad [\Gamma_4,\Gamma_{15}]=i\Gamma_{32};\quad [\Gamma_4,\Gamma_{16}]=\Gamma_{28};\quad [\Gamma_4,\Gamma_{19}]=\Gamma_{33};$$
$$[\Gamma_4,\Gamma_{22}]=i\Gamma_{35};\quad [\Gamma_4,\Gamma_{23}]=i\Gamma_6;\quad [\Gamma_4,\Gamma_{24}]=\Gamma_6;$$
$$[\Gamma_4,\Gamma_{26}]=2i\Gamma_4;\quad [\Gamma_4,\Gamma_{27}]=2\Gamma_4;\quad [\Gamma_4,\Gamma_{29}]=\Gamma_{35};\quad [\Gamma_4,\Gamma_{31}]=i\Gamma_{33};$$
$$[\Gamma_5,\Gamma_7]=i\Gamma_{21}+i\Gamma_{17}+\Gamma_{18}+\Gamma_{20};\quad [\Gamma_5,\Gamma_9]=i\Gamma_{18}-\Gamma_{21};$$
$$[\Gamma_5,\Gamma_{10}]=i\Gamma_{20}+\Gamma_{17};\quad [\Gamma_5,\Gamma_{13}]=i\Gamma_{31}-\Gamma_{22};\quad [\Gamma_5,\Gamma_{14}]=\Gamma_{19}+i\Gamma_{29},$$
$$[\Gamma_5,\Gamma_{15}]=\Gamma_{31}+i\Gamma_{32};\quad [\Gamma_5,\Gamma_{16}]=-\Gamma_{29}+i\Gamma_{19};\quad [\Gamma_5,\Gamma_{17}]=i\Gamma_5+\Gamma_1,$$
$$[\Gamma_5,\Gamma_{18}]=-\Gamma_5+2i\Gamma_1;\quad [\Gamma_5,\Gamma_{20}]=2i\Gamma_2+\Gamma_5;$$
$$[\Gamma_5,\Gamma_{21}]=\Gamma_{18};\quad [\Gamma_5,\Gamma_{25}]=i\Gamma_{20},\quad [\Gamma_5,\Gamma_{28}]=\Gamma_{35}+i\Gamma_{33};\quad [\Gamma_5,\Gamma_{30}]=-\Gamma_{36}+i\Gamma_{34};$$
$$[\Gamma_5,\Gamma_{32}]=\Gamma_{33}+i\Gamma_{33},\quad [\Gamma_6,\Gamma_8]=i\Gamma_{26}+\Gamma_{23}-\Gamma_{27}+i\Gamma_{24};\quad [\Gamma_6,\Gamma_{11}]=i\Gamma_{23}-\Gamma_{26},$$
$$[\Gamma_6,\Gamma_{12}]=\Gamma_{24}+i\Gamma_{24};\quad [\Gamma_6,\Gamma_{13}]=-\Gamma_{28}+i\Gamma_{30};\quad [\Gamma_6,\Gamma_{14}]=i\Gamma_{32}+\Gamma_{25},$$
$$[\Gamma_6,\Gamma_{15}]=-\Gamma_{32}+i\Gamma_{25};\quad [\Gamma_6,\Gamma_{16}]=i\Gamma_{28}+\Gamma_{30}\quad [\Gamma_6,\Gamma_{19}]=i\Gamma_{33}+\Gamma_{34};\quad [\Gamma_6,\Gamma_{22}]=-\Gamma_{35}+i\Gamma_{36};$$
$$[\Gamma_6,\Gamma_{23}]=-\Gamma_6+2i\Gamma_3;\quad [\Gamma_6,\Gamma_{24}]=2\Gamma_3+i\Gamma_6;\quad [\Gamma_6,\Gamma_{26}]=2\Gamma_3-i\Gamma_4$$
$$[\Gamma_6,\Gamma_{27}]=\Gamma_6+2i\Gamma_4;\quad [\Gamma_6,\Gamma_{29}]=i\Gamma_{35}+\Gamma_{33},[\Gamma_6,\Gamma_{31}]=-\Gamma_{33}+i\Gamma_{34};$$
$$[\Gamma_7,\Gamma_{17}]=-i\Gamma_7+2i\Gamma_{10};\quad [\Gamma_7,\Gamma_{18}]=\Gamma_7-2i\Gamma_9;\quad [\Gamma_7,\Gamma_{19}]=-i\Gamma_{16}-\Gamma_{14};$$
$$[\Gamma_7,\Gamma_{20}]=-\Gamma_7-2i\Gamma_{10};\quad [\Gamma_7,\Gamma_{21}]=-i\Gamma_7-2\Gamma_9;\quad [\Gamma_7,\Gamma_{22}]=i\Gamma_{15}-\Gamma_{13};$$
$$[\Gamma_7,\Gamma_{29}]=-i\Gamma_{14}-\Gamma_{16};\quad [\Gamma_7,\Gamma_{31}]=\Gamma_{15}-i\Gamma_{13};\quad [\Gamma_7,\Gamma_{33}]=-i\Gamma_{28}+\Gamma_{32};$$
$$[\Gamma_7,\Gamma_{34}]=-i\Gamma_{30}+\Gamma_{25};\quad [\Gamma_7,\Gamma_{35}]=-\Gamma_{32}-\Gamma_{28};\quad [\Gamma_7,\Gamma_{36}]=-i\Gamma_{25}-\Gamma_{30};$$
$$[\Gamma_8,\Gamma_{23}]=-2i\Gamma_{11}+\Gamma_8;\quad [\Gamma_8,\Gamma_{24}]=-i\Gamma_8+2\Gamma_{12};\quad [\Gamma_8,\Gamma_{25}]=\Gamma_{14}-i\Gamma_{15},$$
$$[\Gamma_8,\Gamma_{26}]=-2\Gamma_{11}-i\Gamma_8;\quad [\Gamma_8,\Gamma_{27}]=-\Gamma_8-2i\Gamma_{12};\quad [\Gamma_8,\Gamma_{28}]=-\Gamma_{13}-i\Gamma_{16},$$
$$[\Gamma_8,\Gamma_{30}]=-i\Gamma_{13}+\Gamma_{16};\quad [\Gamma_8,\Gamma_{32}]=-\Gamma_{15}-i\Gamma_{14};\quad [\Gamma_8,\Gamma_{33}]=-\Gamma_{31}-i\Gamma_{19},$$
$$[\Gamma_8,\Gamma_{34}]=-i\Gamma_{31}+\Gamma_{19};\quad [\Gamma_8,\Gamma_{35}]=-\Gamma_{22}-i\Gamma_{29};\quad [\Gamma_8,\Gamma_{36}]=-\Gamma_{30}-i\Gamma_{25},$$
$$[\Gamma_9,\Gamma_{17}]=\Gamma_7;\quad [\Gamma_9,\Gamma_{18}]=2\Gamma_9;\quad [\Gamma_9,\Gamma_{19}]=-\Gamma_{16};$$
$$[\Gamma_9,\Gamma_{20}]=-i\Gamma_7;\quad [\Gamma_9,\Gamma_{21}]=-2i\Gamma_9;\quad [\Gamma_9,\Gamma_{22}]=-i\Gamma_3;$$
$$[\Gamma_9,\Gamma_{29}]=-i\Gamma_{16};\quad [\Gamma_9,\Gamma_{31}]=\Gamma_{13};\quad [\Gamma_9,\Gamma_{33}]=\Gamma_{28};$$
$$[\Gamma_9,\Gamma_{34}]=\Gamma_{30};\quad [\Gamma_9,\Gamma_{35}]=-i\Gamma_{28};\quad [\Gamma_9,\Gamma_{36}]=-i\Gamma_{30};$$
$$[\Gamma_{10},\Gamma_{17}]=-2i\Gamma_{10};\quad [\Gamma_{10},\Gamma_{18}]=-i\Gamma_7;\quad [\Gamma_{10},\Gamma_{19}]=-i\Gamma_{14};$$
$$[\Gamma_{10},\Gamma_{20}]=-2\Gamma_{10};\quad [\Gamma_{10},\Gamma_{21}]=-\Gamma_7;\quad [\Gamma_{10},\Gamma_{22}]=-\Gamma_{15};$$
$$[\Gamma_{10},\Gamma_{29}]=-\Gamma_{14};\quad [\Gamma_{10},\Gamma_{31}]=-i\Gamma_{15};\quad [\Gamma_{10},\Gamma_{33}]=-i\Gamma_{32};$$
$$[\Gamma_{10},\Gamma_{34}]=-i\Gamma_{25};\quad [\Gamma_{10},\Gamma_{35}]=-\Gamma_{32};\quad [\Gamma_{10},\Gamma_{36}]=-\Gamma_{25};$$
$$[\Gamma_{11},\Gamma_{23}]=2\Gamma_{11};\quad [\Gamma_{11},\Gamma_{24}]=\Gamma_8;\quad [\Gamma_{11},\Gamma_{25}]=\Gamma_{15},$$

$$
\begin{aligned}
&[\Gamma_{11},\Gamma_{26}] = -2i\Gamma_{11};\quad [\Gamma_{11},\Gamma_{27}] = -i\Gamma_{8};\quad [\Gamma_{11},\Gamma_{28}] = -i\Gamma_{13};\\
&[\Gamma_{11},\Gamma_{30}] = \Gamma_{13};\quad [\Gamma_{11},\Gamma_{32}] = -i\Gamma_{15}; [\Gamma_{11},\Gamma_{33}] = -\Gamma_{19},\\
&[\Gamma_{11},\Gamma_{34}] = \Gamma_{31};\quad [\Gamma_{11},\Gamma_{35}] = -i\Gamma_{22};\quad [\Gamma_{11},\Gamma_{36}] = \Gamma_{22},\\
&[\Gamma_{12},\Gamma_{23}] = -i\Gamma_{8};\quad [\Gamma_{12},\Gamma_{24}] = -2i\Gamma_{12};\quad [\Gamma_{12},\Gamma_{25}] = -i\Gamma_{14},\\
&[\Gamma_{12},\Gamma_{26}] = -\Gamma_{8};\quad [\Gamma_{12},\Gamma_{27}] = -2\Gamma_{12};\quad [\Gamma_{12},\Gamma_{28}] = -\Gamma_{16};\\
&[\Gamma_{12},\Gamma_{30}] = -i\Gamma_{16};\quad [\Gamma_{12},\Gamma_{32}] = -\Gamma_{14}; [\Gamma_{12},\Gamma_{33}] = -\Gamma_{19},\\
&[\Gamma_{12},\Gamma_{34}] = -i\Gamma_{19};\quad [\Gamma_{12},\Gamma_{35}] = -\Gamma_{29};\quad [\Gamma_{12},\Gamma_{36}] = -i\Gamma_{29},\\
&[\Gamma_{13},\Gamma_{17}] = \Gamma_{15};\quad [\Gamma_{13},\Gamma_{18}] = \Gamma_{13};\quad [\Gamma_{13},\Gamma_{19}] = \Gamma_{9};\\
&[\Gamma_{13},\Gamma_{20}] = -i\Gamma_{15};\quad [\Gamma_{13},\Gamma_{21}] = -i\Gamma_{13};\quad [\Gamma_{13},\Gamma_{22}] = -2i\Gamma_{11};\\
&[\Gamma_{13},\Gamma_{23}] = \Gamma_{13};\quad [\Gamma_{13},\Gamma_{24}] = \Gamma_{16};\quad [\Gamma_{13},\Gamma_{25}] = \Gamma_{7};\\
&[\Gamma_{13},\Gamma_{26}] = -i\Gamma_{13};\quad [\Gamma_{13},\Gamma_{27}] = -i\Gamma_{16};\quad [\Gamma_{13},\Gamma_{28}] = -2i\Gamma_{9};\\
&[\Gamma_{13},\Gamma_{29}] = -i\Gamma_{8};\quad [\Gamma_{13},\Gamma_{30}] = 2\Gamma_{9};\quad [\Gamma_{13},\Gamma_{31}] = 2\Gamma_{11};\\
&[\Gamma_{13},\Gamma_{32}] = -i\Gamma_{7};\quad [\Gamma_{13},\Gamma_{33}] = \Gamma_{26} - i\Gamma_{18};\quad [\Gamma_{13},\Gamma_{34}] = \Gamma_{23} + \Gamma_{18};\\
&[\Gamma_{13},\Gamma_{35}] = -i\Gamma_{26} - i\Gamma_{21};\quad [\Gamma_{13},\Gamma_{36}] = \Gamma_{21} - i\Gamma_{13};\\
&[\Gamma_{14},\Gamma_{17}] = -\Gamma_{16};\quad [\Gamma_{14},\Gamma_{18}] = -i\Gamma_{16};\quad [\Gamma_{14},\Gamma_{19}] = -2i\Gamma_{12};\\
&[\Gamma_{14},\Gamma_{20}] = -\Gamma_{14};\quad [\Gamma_{14},\Gamma_{21}] = -\Gamma_{16};\quad [\Gamma_{14},\Gamma_{22}] = -\Gamma_{8};\\
&[\Gamma_{14},\Gamma_{23}] = -i\Gamma_{15};\quad [\Gamma_{14},\Gamma_{24}] = -i\Gamma_{14};\quad [\Gamma_{14},\Gamma_{25}] = -2i\Gamma_{10};\\
&[\Gamma_{14},\Gamma_{26}] = -\Gamma_{15};\quad [\Gamma_{14},\Gamma_{27}] = -\Gamma_{14};\quad [\Gamma_{14},\Gamma_{28}] = -\Gamma_{7};\\
&[\Gamma_{14},\Gamma_{29}] = -2\Gamma_{12};\quad [\Gamma_{14},\Gamma_{30}] = -i\Gamma_{7};\quad [\Gamma_{14},\Gamma_{31}] = -i\Gamma_{25};\\
&[\Gamma_{14},\Gamma_{32}] = -2\Gamma_{10};\quad [\Gamma_{14},\Gamma_{33}] = -i\Gamma_{24} - \Gamma_{17};\quad [\Gamma_{14},\Gamma_{34}] = -i\Gamma_{17} - i\Gamma_{24};\\
&[\Gamma_{14},\Gamma_{35}] = -2\Gamma_{20} - \Gamma_{27};\quad [\Gamma_{14},\Gamma_{36}] = -\Gamma_{20} - i\Gamma_{24};\\
&[\Gamma_{15},\Gamma_{17}] = -i\Gamma_{15};\quad [\Gamma_{15},\Gamma_{18}] = -i\Gamma_{13};\quad [\Gamma_{15},\Gamma_{19}] = -i\Gamma_{8};\\
&[\Gamma_{15},\Gamma_{20}] = -\Gamma_{20};\quad [\Gamma_{15},\Gamma_{21}] = -\Gamma_{13};\quad [\Gamma_{15},\Gamma_{22}] = -2\Gamma_{11};\\
&[\Gamma_{15},\Gamma_{23}] = \Gamma_{15};\quad [\Gamma_{15},\Gamma_{24}] = \Gamma_{14};\quad [\Gamma_{15},\Gamma_{25}] = 2\Gamma_{10};\\
&[\Gamma_{15},\Gamma_{26}] = -i\Gamma_{15};\quad [\Gamma_{15},\Gamma_{27}] = -i\Gamma_{14};\quad [\Gamma_{15},\Gamma_{28}] = -i\Gamma_{27};\\
&[\Gamma_{15},\Gamma_{29}] = -\Gamma_{8};\quad [\Gamma_{15},\Gamma_{30}] = \Gamma_{7};\quad [\Gamma_{15},\Gamma_{31}] = -2i\Gamma_{11};\\
&[\Gamma_{15},\Gamma_{32}] = -2i\Gamma_{10};\quad [\Gamma_{15},\Gamma_{33}] = -i\Gamma_{26} - i\Gamma_{17};\quad [\Gamma_{15},\Gamma_{34}] = \Gamma_{17} - i\Gamma_{14};\\
&[\Gamma_{15},\Gamma_{35}] = -\Gamma_{26} - i\Gamma_{20};\quad [\Gamma_{15},\Gamma_{36}] = \Gamma_{20} - \Gamma_{23};\\
&[\Gamma_{16},\Gamma_{17}] = \Gamma_{14};\quad [\Gamma_{16},\Gamma_{18}] = \Gamma_{16};\quad [\Gamma_{16},\Gamma_{19}] = 2\Gamma_{12};\\
&[\Gamma_{16},\Gamma_{20}] = -i\Gamma_{14};\quad [\Gamma_{16},\Gamma_{21}] = -i\Gamma_{16};\quad [\Gamma_{16},\Gamma_{22}] = i\Gamma_{8};\\
&[\Gamma_{16},\Gamma_{23}] = -i\Gamma_{13};\quad [\Gamma_{16},\Gamma_{24}] = -i\Gamma_{16};\quad [\Gamma_{16},\Gamma_{25}] = -i\Gamma_{7};\\
&[\Gamma_{16},\Gamma_{26}] = -\Gamma_{13};\quad [\Gamma_{16},\Gamma_{27}] = -\Gamma_{17};\quad [\Gamma_{16},\Gamma_{28}] = -2\Gamma_{9};\\
&[\Gamma_{16},\Gamma_{29}] = -2i\Gamma_{12};\quad [\Gamma_{16},\Gamma_{30}] = -2i\Gamma_{9};\quad [\Gamma_{16},\Gamma_{31}] = \Gamma_{8};\\
&[\Gamma_{16},\Gamma_{32}] = -\Gamma_{7};\quad [\Gamma_{16},\Gamma_{33}] = -\Gamma_{18} + \Gamma_{27};\quad [\Gamma_{16},\Gamma_{34}] = \Gamma_{24} - i\Gamma_{18};\\
&[\Gamma_{16},\Gamma_{35}] = -\Gamma_{21} - i\Gamma_{27};\quad [\Gamma_{16},\Gamma_{36}] = -i\Gamma_{24} - i\Gamma_{21};
\end{aligned} \tag{3.102}
$$

Therefore, their use in eq.(3.66) yields the following set of 36 PDEs in λ's:

$$\dot{\lambda}_1 = 4(i\lambda_{17} - \lambda_{18}), \tag{3.103}$$

$$\dot{\lambda}_2 = -4(\lambda_{20} + i\lambda_{21}), \tag{3.104}$$

$$\dot{\lambda}_3 = -4(\lambda_{23} - i\lambda_{24}), \tag{3.105}$$

$$\dot{\lambda}_4 = -4(i\lambda_{26} + \lambda_{27}), \tag{3.106}$$

$$\dot{\lambda}_5 = -2(\lambda_{17} + i\lambda_{18} + \lambda_{21} - i\lambda_{20}), \tag{3.107}$$

$$\dot{\lambda}_6 = -2(i\lambda_{23} - \lambda_{24} - i\lambda_{27} + \lambda_{26}), \tag{3.108}$$

$$\dot{\lambda}_7 = 2\kappa_2(\lambda_{17} + i\lambda_{18} + \lambda_{21} + i\lambda_{20}) + 2\kappa_1(\lambda_{32} + i\lambda_{30} + \lambda_{28} - i\lambda_{25}), \tag{3.109}$$

$$\dot{\lambda}_8 = 2\kappa_1(\lambda_{22} - i\lambda_{19} - i\lambda_{29} + i\lambda_{31}) + 2\kappa_2(i\lambda_{23} + \lambda_{24} - i\lambda_{27} + \lambda_{26}), \tag{3.110}$$

$$\dot{\lambda}_9 = 4\kappa_2(\lambda_{18} - i\lambda_{21}) + 4\kappa_1(-i\lambda_{28} + \lambda_{30}), \tag{3.111}$$

$$\dot{\lambda}_{10} = 4\kappa_2(i\lambda_{17} - \lambda_{20}) + 4\kappa_1(i\lambda_{32} - \lambda_{25}), \tag{3.112}$$

$$\dot{\lambda}_{11} = 4\kappa_2(\lambda_{23} - i\lambda_{26}) + 4\kappa_1(\lambda_{31} - i\lambda_{22}), \tag{3.113}$$

$$\dot{\lambda}_{12} = 4\kappa_2(i\lambda_{24} + \lambda_{27}) + 4\kappa_1(\lambda_{19} + \lambda_{29}), \tag{3.114}$$

$$\dot{\lambda}_{13} = 2\kappa_2(-i\lambda_{22} - i\lambda_{28} + \lambda_{30} + \lambda_{31}) + 2\kappa_1(\lambda_{18} - i\lambda_{21} + \lambda_{23} - i\lambda_{26}), \tag{3.115}$$

$$\dot{\lambda}_{14} = 2\kappa_2(\lambda_{19} - \lambda_{25} + \lambda_{27} + i\lambda_{32}) + 2\kappa_1(i\lambda_{17} - \lambda_{20} + \lambda_{27} + i\lambda_{24}), \tag{3.116}$$

$$\dot{\lambda}_{15} = 2\kappa_2(\lambda_{22} - i\lambda_{25} - \lambda_{32} + i\lambda_{31}) + 2\kappa_1(-\lambda_{17} - i\lambda_{20} + \lambda_{26} + i\lambda_{23}), \tag{3.117}$$

$$\dot{\lambda}_{16} = 2\kappa_2(i\lambda_{19} + \lambda_{28} + i\lambda_{29} + i\lambda_{30}) + 2\kappa_1(i\lambda_{18} + \lambda_{21} - \lambda_{24} + i\lambda_{27}), \tag{3.118}$$

$$\dot{\lambda}_{17} = 2\kappa_2(i\lambda_1 + \lambda_5) + 2(-\lambda_{10} - \lambda_7) + 2\kappa_1(\lambda_{33} + i\lambda_{34}), \tag{3.119}$$

$$\dot{\lambda}_{18} = 2\kappa_2(\lambda_1 - i\lambda_5) + 2(i\lambda_7 - i\lambda_9) + 2\kappa_1(-i\lambda_{33} + \lambda_{34}), \tag{3.120}$$

$$\dot{\lambda}_{19} = 2\kappa_2(\lambda_{33} + i\lambda_{34}) + 2(-i\lambda_{14} - \lambda_{16}) + 2\kappa_1(i\lambda_1 + \lambda_5), \tag{3.121}$$

$$\dot{\lambda}_{20} = 2\kappa_2(i\lambda_2 - i\lambda_5) + 2(i\lambda_7 - i\lambda_{10}) + 2\kappa_1(\lambda_{35} - i\lambda_{36}), \tag{3.122}$$

$$\dot{\lambda}_{21} = 2\kappa_2(-\lambda_2 + \lambda_5) + 2(-\lambda_7 - \lambda_9) + 2\kappa_1(i\lambda_{35} + \lambda_{36}), \tag{3.123}$$

$$\dot{\lambda}_{22} = 2\kappa_2(-i\lambda_{35} + \lambda_{36}) + 2(-i\lambda_{13} - \lambda_{15}) + 2\kappa_1(-i\lambda_2 + \lambda_5), \tag{3.124}$$

$$\dot{\lambda}_{23} = 2\kappa_2(\lambda_3 - i\lambda_6) + 2(-\lambda_{11} + i\lambda_8) + 2\kappa_1(\lambda_{34} - i\lambda_{36}), \tag{3.125}$$

$$\dot{\lambda}_{24} = 2\kappa_2(i\lambda_3 + \lambda_6) + 2(-\lambda_8 + i\lambda_{12}) + 2\kappa_1(\lambda_{36} + i\lambda_{34}), \tag{3.126}$$

$$\dot{\lambda}_{25} = 2\kappa_2(\lambda_{34} - i\lambda_{36}) + 2(-\lambda_{15} + i\lambda_{14}) + 2\kappa_1(\lambda_3 + \lambda_6), \tag{3.127}$$

$$\dot{\lambda}_{26} = 2\kappa_2(-i\lambda_4 + \lambda_6) + 2(-\lambda_8 - i\lambda_{11}) + 2\kappa_1(\lambda_{33} - i\lambda_{35}), \tag{3.128}$$

$$\dot{\lambda}_{27} = 2\kappa_2(\lambda_4 + i\lambda_6) + 2(-\lambda_{12} - i\lambda_8) + 2\kappa_1(\lambda_{35} + i\lambda_{33}), \tag{3.129}$$

$$\dot{\lambda}_{28} = 2\kappa_2(\lambda_{33} - i\lambda_{35}) + 2(-i\lambda_{13} - \lambda_{16}) + 2\kappa_1(\lambda_6 - i\lambda_4), \tag{3.130}$$

$$\dot{\lambda}_{29} = 2\kappa_2(\lambda_{35} + i\lambda_{36}) + 2(\lambda_{14} - i\lambda_{16}) + 2\kappa_1(\lambda_2 + i\lambda_5), \tag{3.131}$$

$$\dot{\lambda}_{30} = 2\kappa_2(\lambda_{34} - i\lambda_{36}) + 2(-\lambda_{13} + i\lambda_{16}) + 2\kappa_1(\lambda_3 + i\lambda_6), \tag{3.132}$$

$$\dot{\lambda}_{31} = 2\kappa_2(-i\lambda_{33} + \lambda_{34}) + 2(-\lambda_{13} + i\lambda_{15}) + 2\kappa_1(\lambda_1 - i\lambda_5), \tag{3.133}$$

$$\dot{\lambda}_{32} = 2\kappa_2(-i\lambda_{33} + \lambda_{35}) + 2(+\lambda_{14} - i\lambda_{15}) + 2\kappa_1(\lambda_4 + i\lambda_6), \tag{3.134}$$

$$\dot{\lambda}_{33} = 2(-\lambda_{19} - i\lambda_{31} - \lambda_{28} + i\lambda_{32}), \tag{3.135}$$

$$\dot{\lambda}_{34} = 2(i\lambda_{19} - \lambda_{30} - \lambda_{31} + \lambda_{25}), \tag{3.136}$$

$$\dot{\lambda}_{35} = 2(-i\lambda_{22} - i\lambda_{28} - \lambda_{32} - \lambda_{29}), \tag{3.137}$$

$$\dot{\lambda}_{36} = 2(-\lambda_{22} - \lambda_{25} + i\lambda_{29} - i\lambda_{30}), \tag{3.138}$$

In fact, to solve these 36 coupled PDEs for complex λ's is very difficult. Thus, here we make certain choices for λ's which facilitate to find solutions of above equations.

From eqs.(3.103), (3.104) and (3.107), we get $2\dot{\lambda}_5 = i\dot{\lambda}_1 - i\dot{\lambda}_2$; and if we consider $\lambda_5 = c_5$ (a constant), and from relation $\lambda_1 = \lambda_2 = \eta_1(t)$, which immediately gives

$$\lambda_1 = \eta_1(t) + c_1; \qquad \lambda_2 = \eta_1(t) + c_2 \tag{3.139}$$

Again from eqs.(3.105), (3.106) and (3.108), we get $2\dot{\lambda}_6 = i\dot{\lambda}_3 + i\dot{\lambda}_4$; and if we consider $\lambda_6 = c_6$ (a constant); and from relation $\lambda_3 = \lambda_4 = \eta_2(t)$; which immediately gives

$$\lambda_3 = \eta_2(t) + c_3; \qquad \lambda_4 = -\eta_2(t) + c_4. \tag{3.140}$$

Now, in order to find solutions for $\lambda_{11}, \lambda_{12}$, From eqs.(3.113), (3.114) and (3.110), we get $2\dot{\lambda}_8 = i\dot{\lambda}_{11} - i\dot{\lambda}_{12}$; and if we consider $\lambda_8 = c_8$ (a constant), and from relation $\lambda_{11} = \lambda_{12} = \eta_3(t)$; which immediately gives

$$\lambda_{11} = \eta_3(t) + c_{11}; \qquad \lambda_{12} = \eta_3(t) + c_{12}. \tag{3.141}$$

From eqs.(3.111), (3.112) and (3.109), we get $2\dot{\lambda}_7 = i\dot{\lambda}_9 - i\dot{\lambda}_{10}$, and if we consider $\lambda_7 = c_7$ (a constant), and from relation $\lambda_9 = \lambda_{10} = \eta_4(t)$, which immediately gives

$$\lambda_9 = \eta_4(t) + c_9; \qquad \lambda_{10} = \eta_4(t) + c_{10}. \tag{3.142}$$

Now, in order to find solutions for λ_{13}, From eqs.(3.111) , (3.113) and (3.115), we get $2\dot{\lambda}_{13} = \dot{\lambda}_9 + \dot{\lambda}_{11}$, and if we consider $\lambda_{13} = c_{13}$ (a constant), from above relation (with $\lambda_9 = \eta_4(t) + c_9; \quad \lambda_{11} = \eta_3(t) + c_{11}$) gives

$$\lambda_{13} = \eta(t) + c_{13}. \tag{3.143}$$

where $\eta(t) = \eta_4(t) + \eta_3(t)$; is an another function of t and ($c_{13} = c_3 + c_4$, a constant)
From eqs.(3.112), (3.113) and (3.117), we get $2\dot{\lambda}_{15} = i\dot{\lambda}_{10} + i\dot{\lambda}_{11}$;, and if we consider $\lambda_{15} = c_{15}$ (a constant), further from above relation (with $\lambda_{10} = \eta_4(t) + c_{10}; \quad \lambda_{11} = \eta_3(t) + c_{11}$) gives

$$\lambda_{15} = \eta(t) + c_{15}. \tag{3.144}$$

where $\eta(t) = \eta_4(t) + \eta_3(t)$ is an another function of t and ($c_{15} = c_{10} + c_{11}$, a constant)
Again, in order to find solutions for λ_{16}, From eqs.(3.111), (3.114) and (3.118), we get $2\dot{\lambda}_{16} = i\dot{\lambda}_9 + i\dot{\lambda}_{12}$, and if we consider $\lambda_{16} = c_{16}$ (a constant), further from above relation (with $\lambda_9 = \eta_4(t) + c_9; \quad \lambda_{12} = \eta_3(t) + c_{12}$) gives

$$\lambda_{16} = \eta(t) + c_{16}. \tag{3.145}$$

where $\eta(t) = \eta_4(t) + \eta_3(t)$ is an another function of t and ($c_{16} = c_9 + c_{12}$, a constant)
From eqs.(3.112) , (3.114) and (3.116), we get $2\dot{\lambda}_{14} = \dot{\lambda}_{10} + \dot{\lambda}_{12}$, and if we consider $\lambda_{14} = c_{14}$ (a constant), further from above relation (with $\lambda_{10} = \eta_4(t) + c_{10}; \quad \lambda_{12} = \eta_3(t) + c_{12}$) gives

$$\lambda_{14} = \eta(t) + c_{14}. \tag{3.146}$$

where $\eta(t) = \eta_4(t) + \eta_3(t)$ is an another function of t and ($c_{16} = c_{10} + c_{12}$, a constant)
Now for finding the solutions of λ_{17} and λ_{18}, add i times eq.(3.119) from eq.(3.120) and after using eq.(3.126), we get

$$\dot{\lambda}_{18} + i\dot{\lambda}_{17} = -i\lambda_9 - i\lambda_{10} = -i(\lambda_9 + \lambda_{10}),$$

or

$$i\dot{\lambda}_{17} + \dot{\lambda}_{18} = -i(2\eta_4(t) + c_9 + c_{10}). \tag{3.147}$$

On the other hand, time derivative of eq.(3.103) is written as

$$\ddot{\lambda}_1 = 4(i\lambda_{17} - \lambda_{18}) = \ddot{\theta}. \tag{3.148}$$

Hence using eqs.(3.147) and (3.148), one immediately get

$$\lambda_{17} = -\frac{i}{8}(\dot{\theta} - 8\xi) + c_{17}, \tag{3.149}$$

and

$$\lambda_{18} = -\frac{1}{8}(\dot{\theta} + 8\xi) + c_{18}, \tag{3.150}$$

where $\xi = \int(2\eta_4(t) + c_9 + c_{10})dt$.

Similarly, from eqs.(3.122 to 3.129) with eq.(3.106), we obtain solutions for $(\lambda_{20} - \lambda_{27})$ respectively as

$$\lambda_{20} = -\frac{1}{8}(\dot{\theta} + 8\xi) + c_{20}; \qquad \lambda_{21} = \frac{i}{8}(\dot{\theta} - 8\xi) + c_{21}, \tag{3.151}$$

$$\lambda_{23} = -\frac{1}{8}(\dot{\rho} - 8i\sigma) + c_{23}; \qquad \lambda_{24} = -\frac{i}{8}(\dot{\rho} + 8i\sigma) + c_{24}, \tag{3.152}$$

$$\lambda_{26} = \frac{i}{8}(\dot{\rho} + 8i\sigma) + c_{26}; \qquad \lambda_{27} = -\frac{1}{8}(\dot{\rho} - 8i\sigma) + c_{27}, \tag{3.153}$$

where $\sigma = \int(2i\eta_3(t) + c_{11} + c_{12})dt$. Now to obtain solutions for $(\lambda_{19} - \lambda_{32})$ respectively as, from eqs.(3.121 to 3.134), we obtain following equations

$$\dot{\lambda}_{19} - i\dot{\lambda}_{31} = (i\lambda_{13} - i\lambda_{14} + \lambda_{15} - \lambda_{16}) = 0. \tag{3.154}$$

$$\dot{\lambda}_{22} + i\dot{\lambda}_{29} = (-i\lambda_{13} + i\lambda_{14} - \lambda_{15} + \lambda_{16}) = 0. \tag{3.155}$$

$$\dot{\lambda}_{25} - \dot{\lambda}_{30} = (\lambda_{13} + i\lambda_{14} - \lambda_{15} - i\lambda_{16}) = 0. \tag{3.156}$$

$$\dot{\lambda}_{28} + i\dot{\lambda}_{32} = (-\lambda_{13} + i\lambda_{14} + \lambda_{15} - \lambda_{16}) = 0. \tag{3.157}$$

since

$$\lambda_{13} = \lambda_{14} = \lambda_{15} = \lambda_{16} = \eta(t).$$

or if we set

$$\dot{\lambda}_{19} = i\dot{\lambda}_{31} = \dot{\phi}(t); \qquad \dot{\lambda}_{22} = -i\dot{\lambda}_{29} = \dot{\varphi}(t). \tag{3.158}$$

$$\dot{\lambda}_{25} = \dot{\lambda}_{30} = \dot{\chi}(t); \qquad \dot{\lambda}_{28} = -i\dot{\lambda}_{32} = \dot{\psi}(t). \tag{3.159}$$

which immediately gives

$$\lambda_{19} = \phi(t) + c_{19}; \qquad \lambda_{31} = -i\phi(t) + c_{31}; \qquad \lambda_{22} = \varphi(t) + c_{22}; \qquad \lambda_{29} = i\varphi(t) + c'_{29} \tag{3.160}$$

$$\lambda_{25} = \chi(t)c_{25}; \qquad \lambda_{30} = \chi(t) + c_{30}; \quad \lambda_{28} = \psi(t) + c_{28}; \qquad \lambda_{32} = i\psi(t) + c_{32}. \tag{3.161}$$

Now to obtain solutions for (λ_{33} to λ_{36}), simply put the values of $(\lambda_{19}, \lambda_{22})$, $(\lambda_{25}, \lambda_{28})$, $(\lambda_{29}, \lambda_{30})$, and $(\lambda_{31}, \lambda_{32})$ in (3.135 to 3.138), we get

$$\lambda_{33} = \alpha_1(t) + c_{33}; \qquad \lambda_{34} = \alpha_2(t) + c_{34}; \quad \lambda_{35} = \alpha_3(t) + c_{35}; \qquad \lambda_{36} = \alpha_4(t) + c_{36} \tag{3.162}$$

where the α's are as follows

$$\begin{aligned}
\alpha_1(t) &= -\int 2[2i\phi(t) - 2i\psi(t) + c_{19} + c_{28} + c_{31} - ic_{32}]dt;\\
\alpha_2(t) &= -\int 2[2\phi(t) - 2i + [\chi(t) - i\chi(t)] - ic_{19} + c_{30} + c_{31} - ic_{25}]dt;\\
\alpha_3(t) &= -\int 2[2i\varphi(t) + 2\psi(t) - ic_{22} - ic_{28} - c_{29} - c_{32}]dt;\\
\alpha_4(t) &= -\int 2[-2i\varphi(t) + [\chi(t) - i\chi(t)] + c_{22} + c_{25} - ic_{29} - ic_{30}]dt.
\end{aligned}$$

We have solved eqs.[(3.103) to (3.138)] in terms of arbitrary functions η's, $\theta, \xi, \sigma, \psi, \phi, \varphi, \rho, \chi$ and α's and complex constants, c_i's, $(i = 1, ..., 36)$. If one put back these solutions for λ_i, $(i = 1, ..., 36)$ in eqs. [(3.103) to (3.138)], we obtain a number of constraint relations among c_i's, and η's, $\theta, \phi, \xi, \sigma, \psi, \varphi, \rho, \chi$ and α's, which limit the choices of these arbitrary complex quantities. If we set all c_i's equal to zero, then these relations with the equations determining arbitrary functions η's, $\theta, \phi, \varphi, \rho, \chi$ and α's are written as

$$\begin{aligned}
&\ddot{\eta}_4 + 4\dot{\theta} - \dot{\chi} + 4i\eta_4 = 0;\ \ddot{\eta}_3 - 4\dot{\phi} - \dot{\rho} + 4i\eta_4 = 0,\\
&\ddot{\eta} - 2(\dot{\varphi} + \dot{\phi} - \dot{\chi} - \dot{\sigma}) = 0;\ \ddot{\eta} - 2(\dot{\varphi} + \dot{\sigma} - \dot{\chi} - \dot{\xi}) = 0,\\
&\ddot{\psi} + 16(i\dot{\alpha}_1 - \dot{\alpha}_2 - \dot{\eta}_1) = 0;\ \ddot{\psi} + 16(\dot{\alpha}_3 - i\dot{\alpha}_4 + i\dot{\eta}_1) = 0,\\
&\ddot{\varphi} + 16(\dot{\alpha}_2 - i\dot{\alpha}_4 + \dot{\eta}_2) = 0;\ \ddot{\varphi} + 16(i\dot{\alpha}_1 + \dot{\alpha}_4 + \dot{\eta}_2) = 0,\\
&\ddot{\psi} + 2i(i\dot{\alpha}_2 + \dot{\alpha}_1 - i\dot{\eta} - \dot{\eta} + i\dot{\eta}_1) = 0;\ \ddot{\sigma} - 2(i\dot{\alpha}_4 + \dot{\alpha}_3 - i\dot{\eta} + \dot{\eta} + i\dot{\eta}_1) = 0,\\
&\ddot{\sigma} - 2(\dot{\alpha}_3 - i\dot{\alpha}_4 + i\dot{\eta} - \dot{\eta} + \dot{\eta}_2) = 0;\ \ddot{\psi} - 2(i\dot{\alpha}_1 + \dot{\alpha}_2 - i\dot{\eta} + \dot{\eta} + i\dot{\eta}_1) = 0,\\
&2i\phi + 2i\psi = 0;\ 2\phi - 2\chi - i\chi = 0;\ 2\varphi + 2\psi = 0;\ 2i\varphi - \chi - i\chi = 0.
\end{aligned} \tag{3.163}$$

Therefore, after substituting the solutions of λ_i's in the eq.(3.66), the complex invariant for a two dimensional complex oscillator becomes

$$\begin{aligned}
I = &\frac{1}{2}\eta_1(p_1^2 + x_3^2) + \frac{1}{2}\eta_2(p_2^2 + x_4^2) + \frac{1}{2}\eta_4(x_1^2 + p_3^2) + \frac{1}{2}\eta_3(x_2^2 + p_4^2) + \eta(x_1x_2 + p_3p_4 + x_2p_3 + x_1p_4)\\
&-\frac{i}{8}(\dot{\theta} - 8\xi)(p_1p_3 - x_1x_3) - \frac{1}{8}(\dot{\theta} + 8\xi)(x_1p_1 + x_3p_3) - \frac{i}{8}(\dot{\rho} - i8\sigma)(p_2p_4 - x_2x_4) - \frac{1}{8}(\dot{\rho} + i8\sigma)\\
&(p_2x_2 + p_4x_4)\phi p_1p_4 + \varphi x_2x_3 + \psi x_1x_4 + \chi p_2p_3 + i\varphi p_4x_3 + \chi p_2x_1 - i\phi p_1x_2 + i\psi p_3x_4\\
&+\alpha_1p_1x_4 + \alpha_2p_1p_2 + \alpha_3x_3x_4 + \alpha_4p_2x_3.
\end{aligned} \tag{3.164}$$

which conforms to condition eq.(3.66) in view of the Poisson bracket.

3. Coupled harmonic oscillator system

Consider a coupled harmonic oscillator in two-dimensions [18], whose Hamiltonian is given by

$$H = \frac{1}{2}[\alpha_1 p_x^2 + \alpha_2 p_y^2 + \beta_1 x^2 + \beta_2 y^2 + 2\alpha_3 p_x p_y + 2\beta_3 xy]. \tag{3.165}$$

The above Hamiltonian can be expressed as

$$H = \alpha_1\left(\frac{1}{2}p_1^2 - \frac{1}{2}x_3^2 + ip_1x_3\right) + \alpha_2\left(\frac{1}{2}p_2^2 - \frac{1}{2}x_4^2 + ip_2x_4\right) + \beta_1\left(\frac{1}{2}x_1^2 - \frac{1}{2}p_3^2 + ip_3x_1\right) + \beta_2\left(\frac{1}{2}x_2^2 - \frac{1}{2}p_4^2 + ix_2p_4\right) + \alpha_3(x_1x_2 + ix_1p_4 + ip_3x_2 - p_3p_4) + \beta_3(p_1p_2 + ip_1x_4 + ip_2x_3 + x_3x_4) = \sum_{m=1}^{20} h_m(t)\Gamma_m \tag{3.166}$$

and the various Γ's and h(t)'s for the above complex H are given as

$$\Gamma_1 = \frac{p_1^2}{2};\ \Gamma_2 = \frac{x_3^2}{2};\ \Gamma_3 = p_1x_3;\ \Gamma_4 = \frac{p_2^2}{2};\ \Gamma_5 = \frac{x_4^2}{2};\ \Gamma_6 = p_2x_4;\ \Gamma_7 = \frac{x_1^2}{2};\ \Gamma_8 = \frac{p_3^2}{2};$$
$$\Gamma_9 = x_1p_3;\ \Gamma_{10} = \frac{x_2^2}{2};\ \Gamma_{11} = x_2p_4;\ \Gamma_{12} = \frac{p_4^2}{2};\ \Gamma_{13} = x_1x_3;\ \Gamma_{14} = x_1p_4;\ \Gamma_{15} = p_3x_2;$$
$$\Gamma_{16} = p_3p_4,\quad \Gamma_{17} = p_1p_2;\quad \Gamma_{18} = p_1x_4;\quad \Gamma_{19} = x_3p_2;\quad \Gamma_{20} = x_3x_4. \tag{3.167}$$

with

$$h_1 = h_2 = \alpha_1;\ h_3 = i\alpha_1;\ h_4 = h_5 = \alpha_2;\ h_6 = i\alpha_2;\ h_7 = h_8 = \beta_1;\ h_9 = i\beta_1;\ h_{10} = h_{11} = \beta_2$$
$$h_{12} = i\beta_2;\ h_{14} = h_{15} = i\beta_3;\ h_{13} = \beta_3;\ h_{16} = -\beta_3;\ h_{17} = \alpha_3;\ h_{18} = h_{19} = i\alpha_3;\ h_{20} = -\alpha_3. \tag{3.168}$$

The dynamical algebra in this case is not closed. To find closure property for the above system, we add sixteen more phase space functions (Γ_l)'s. The additional (Γ_l)'s are as follow

$$\Gamma_{21} = p_1p_3;\quad \Gamma_{22} = p_1x_1;\quad \Gamma_{23} = x_2p_1;\quad \Gamma_{24} = p_1p_4,$$
$$\Gamma_{25} = p_3x_3;\quad \Gamma_{26} = x_1x_3;\quad \Gamma_{27} = x_2x_3;\quad \Gamma_{28} = x_3p_4;$$
$$\Gamma_{29} = x_2p_2;\quad \Gamma_{30} = p_4p_2;\quad \Gamma_{31} = p_2p_3;\quad \Gamma_{32} = p_2x_1,$$
$$\Gamma_{33} = x_2x_4;\quad \Gamma_{34} = x_4p_4;\quad \Gamma_{35} = x_1x_4;\quad \Gamma_{36} = p_3x_4 \tag{3.169}$$

with corresponding $h_l(t) = 0$. Now in the light of Poisson bracket for complex systems, eq.(3.67), we get large number of non-vanishing Poisson brackets, namely

$$[\Gamma_1,\Gamma_7] = -\Gamma_{22};\ \ [\Gamma_1,\Gamma_8] = i\Gamma_{21};\ \ [\Gamma_1,\Gamma_9] = i\Gamma_{22} - \Gamma_{21},\ \ [\Gamma_1,\Gamma_{13}] = -\Gamma_{23};$$
$$[\Gamma_1,\Gamma_{14}] = -\Gamma_{24};\ \ [\Gamma_1,\Gamma_{15}] = i\Gamma_{23};\ \ [\Gamma_1,\Gamma_{16}] = i\Gamma_{24};\ \ [\Gamma_1,\Gamma_{21}] = 2i\Gamma_1;$$
$$[\Gamma_1,\Gamma_{22}] = -2\Gamma_1,\ \ [\Gamma_1,\Gamma_{25}] = i\Gamma_3;\ \ [\Gamma_1,\Gamma_{26}] = -\Gamma_3;\ \ [\Gamma_1,\Gamma_{31}] = i\Gamma_{17};$$
$$[\Gamma_1,\Gamma_{32}] = -\Gamma_{17};\ \ [\Gamma_1,\Gamma_{35}] = i\Gamma_{18};\ \ [\Gamma_1,\Gamma_{36}] = -\Gamma_{18};\ \ [\Gamma_2,\Gamma_7] = i\Gamma_{26}, [\Gamma_2,\Gamma_8] = \Gamma_{25}$$
$$[\Gamma_2,\Gamma_9] = i\Gamma_{25} + i\Gamma_{26};\ \ [\Gamma_2,\Gamma_{13}] = i\Gamma_{27};\ \ [\Gamma_2,\Gamma_{14}] = i\Gamma_{28},$$
$$[\Gamma_2,\Gamma_{15}] = \Gamma_{27};\ \ [\Gamma_2,\Gamma_{16}] = \Gamma_{28};\ \ [\Gamma_2,\Gamma_{21}] = \Gamma_3,\ \ [\Gamma_2,\Gamma_{22}] = i\Gamma_3$$
$$[\Gamma_2,\Gamma_{25}] = 2\Gamma_3;\ \ [\Gamma_2,\Gamma_{26}] = 2i\Gamma_2;\ \ [\Gamma_2,\Gamma_{32}] = i\Gamma_9,$$
$$[\Gamma_2,\Gamma_{35}] = i\Gamma_{20};\ \ [\Gamma_2,\Gamma_{36}] = \Gamma_{20};\ \ [\Gamma_3,\Gamma_7] = -i\Gamma_{26} + i\Gamma_{22};\ [\Gamma_3,\Gamma_8] = i\Gamma_{25} + \Gamma_{21},$$
$$[\Gamma_3,\Gamma_9] = \Gamma_{25} + i\Gamma_{26} + i\Gamma_{21} + \Gamma_{22}\ \ [\Gamma_3,\Gamma_{13}] = -\Gamma_{27} + i\Gamma_{23},$$
$$[\Gamma_3,\Gamma_{14}] = i\Gamma_{24} - \Gamma_{28};\ \ [\Gamma_3,\Gamma_{15}] = i\Gamma_{27} + \Gamma_{23};\ \ [\Gamma_3,\Gamma_{16}] = i\Gamma_{28} + \Gamma_{24};\ \ [\Gamma_3,\Gamma_{21}] = i\Gamma_3 + 2\Gamma_1,$$
$$[\Gamma_3,\Gamma_{22}] = -\Gamma_3 + 2i\Gamma_6;\ \ [\Gamma_3,\Gamma_{25}] = 2i\Gamma_2 + \Gamma_3;\ [\Gamma_3,\Gamma_{26}] = -2i\Gamma_2 + i\Gamma_3$$
$$,\quad [\Gamma_3,\Gamma_{31}] = \Gamma_{17} + i\Gamma_{19},\ \ [\Gamma_3,\Gamma_{32}] = -\Gamma_{19} + i\Gamma_{17},$$
$$[\Gamma_3,\Gamma_{35}] = i\Gamma_{18} - \Gamma_{20};\ \ [\Gamma_3,\Gamma_{36}] = i\Gamma_{20} + i\Gamma_{18};\ \ [\Gamma_4,\Gamma_{10}] = -\Gamma_{29};$$
$$[\Gamma_4,\Gamma_{11}] = -\Gamma_{30} + i\Gamma_{29};\ \ [\Gamma_4,\Gamma_{12}] = i\Gamma_{30};$$
$$[\Gamma_4,\Gamma_{13}] = -\Gamma_{32};\ \ [\Gamma_4,\Gamma_{14}] = i\Gamma_{32};\ \ [\Gamma_4,\Gamma_{15}] = -i\Gamma_{31};\ \ [\Gamma_4,\Gamma_{16}] = i\Gamma_{32},$$

$[\Gamma_4,\Gamma_{23}]=-\Gamma_{17};\quad [\Gamma_4,\Gamma_{24}]=i\Gamma_{17};\quad [\Gamma_4,\Gamma_{27}]=-\Gamma_{19};$

$[\Gamma_4,\Gamma_{28}]=i\Gamma_{19};\quad [\Gamma_4,\Gamma_{29}]=-2\Gamma_4;\quad [\Gamma_4,\Gamma_{30}]=2i\Gamma_4;\quad [\Gamma_4,\Gamma_{33}]=-\Gamma_6,\quad [\Gamma_4,\Gamma_{34}]=i\Gamma_6$

$[\Gamma_5,\Gamma_{10}]=i\Gamma_{33};\quad [\Gamma_5,\Gamma_{11}]=i\Gamma_{34}+\Gamma_{33};\quad [\Gamma_5,\Gamma_{12}]=\Gamma_{34},$

$[\Gamma_5,\Gamma_{13}]=i\Gamma_{35};\quad [\Gamma_5,\Gamma_{14}]=\Gamma_{35};\quad [\Gamma_5,\Gamma_{15}]=i\Gamma_{36},$

$[\Gamma_5,\Gamma_{16}]=\Gamma_{36};\quad [\Gamma_5,\Gamma_{23}]=i\Gamma_{28};\quad [\Gamma_5,\Gamma_{24}]=\Gamma_{18};\quad [\Gamma_5,\Gamma_{27}]=i\Gamma_{20},$

$[\Gamma_5,\Gamma_{28}]=\Gamma_{20};\quad [\Gamma_5,\Gamma_{29}]=i\Gamma_6;\quad [\Gamma_5,\Gamma_{30}]=i\Gamma_6;\quad [\Gamma_5,\Gamma_{33}]=2i\Gamma_5;\quad [\Gamma_5,\Gamma_{34}]=2\Gamma_5,$

$[\Gamma_6,\Gamma_8]=-\Gamma_{33}+i\Gamma_{29};\quad [\Gamma_6,\Gamma_{11}]=i\Gamma_{33}+\Gamma_{29}+i\Gamma_{30}-\Gamma_{34},$

$[\Gamma_6,\Gamma_{12}]=\Gamma_{30}+i\Gamma_{34};\quad [\Gamma_6,\Gamma_{13}]=-\Gamma_{35}+i\Gamma_{32};\quad [\Gamma_6,\Gamma_{14}]=i\Gamma_{35}+\Gamma_{32},$

$[\Gamma_6,\Gamma_{15}]=-\Gamma_{36}+i\Gamma_{31};\quad [\Gamma_6,\Gamma_{16}]=i\Gamma_{36}+\Gamma_{31};$

$[\Gamma_6,\Gamma_{23}]=-\Gamma_{18}+i\Gamma_{17};\quad [\Gamma_6,\Gamma_{24}]=\Gamma_{17}+i\Gamma_{18};\quad [\Gamma_6,\Gamma_{27}]=i\Gamma_{29}-\Gamma_{20}$

$[\Gamma_6,\Gamma_{28}]=\Gamma_{19}+i\Gamma_{20};\quad [\Gamma_6,\Gamma_{29}]=2i\Gamma_4-\Gamma_6,\quad [\Gamma_6,\Gamma_{30}]=2\Gamma_4+i\Gamma_6;$

$[\Gamma_6,\Gamma_{33}]=-2\Gamma_5+i\Gamma_6;\quad [\Gamma_6,\Gamma_{34}]=2i\Gamma_5+\Gamma_6;$

$[\Gamma_7,\Gamma_{17}]=-\Gamma_{32};\ [\Gamma_7,\Gamma_{18}]=-\Gamma_{35};\ [\Gamma_7,\Gamma_{19}]=-i\Gamma_{32};\ [\Gamma_7,\Gamma_{20}]=-i\Gamma_{35};\quad [\Gamma_7,\Gamma_{21}]=\Gamma_9;$

$[\Gamma_7,\Gamma_{22}]=-2\Gamma_7;\ [\Gamma_7,\Gamma_{23}]=\Gamma_{13};\quad [\Gamma_7,\Gamma_{24}]=\Gamma_9;\quad [\Gamma_7,\Gamma_{25}]=-2\Gamma_7;$

$[\Gamma_7,\Gamma_{26}]=-2i\Gamma_7;\quad [\Gamma_7,\Gamma_{27}]=-i\Gamma_{13};\quad [\Gamma_7,\Gamma_{28}]=-i\Gamma_{14};$

$[\Gamma_8,\Gamma_{17}]=-i\Gamma_{31};\quad [\Gamma_8,\Gamma_{18}]=-i\Gamma_{36};\quad [\Gamma_8,\Gamma_{19}]=-\Gamma_{31};\quad [\Gamma_8,\Gamma_{20}]=-\Gamma_{36};\quad [\Gamma_8,\Gamma_{21}]=-2i\Gamma_8;$

$[\Gamma_8,\Gamma_{22}]=-i\Gamma_9;\quad [\Gamma_8,\Gamma_{23}]=-i\Gamma_{15};\quad [\Gamma_8,\Gamma_{24}]=-i\Gamma_{16};\quad [\Gamma_8,\Gamma_{25}]=-2\Gamma_8;$

$[\Gamma_8,\Gamma_{26}]=-\Gamma_9;\quad [\Gamma_8,\Gamma_{27}]=-\Gamma_{15};\quad [\Gamma_8,\Gamma_{28}]=-\Gamma_{16};$

$[\Gamma_9,\Gamma_{17}]=\Gamma_{31}-i\Gamma_{32};\quad [\Gamma_9,\Gamma_{18}]=\Gamma_{36}-i\Gamma_{35};\quad [\Gamma_9,\Gamma_{19}]=-\Gamma_{32}-i\Gamma_{31};$

$[\Gamma_9,\Gamma_{20}]=-i\Gamma_{36}-\Gamma_{35};\quad [\Gamma_9,\Gamma_{21}]=2\Gamma_8-i\Gamma_9;\quad [\Gamma_9,\Gamma_{22}]=\Gamma_9-2i\Gamma_7;$

$[\Gamma_9,\Gamma_{23}]=\Gamma_{15}-i\Gamma_{13};\quad [\Gamma_9,\Gamma_{24}]=\Gamma_{16}-i\Gamma_{14};\quad [\Gamma_9,\Gamma_{25}]=-2i\Gamma_8-\Gamma_9;$

$[\Gamma_9,\Gamma_{26}]=-2\Gamma_7-i\Gamma_9;\quad [\Gamma_9,\Gamma_{27}]=-i\Gamma_{15}-\Gamma_{13};\quad [\Gamma_9,\Gamma_{28}]=-i\Gamma_{16}-\Gamma_{14};$

$[\Gamma_{10},\Gamma_{17}]=\Gamma_{23};\quad [\Gamma_{10},\Gamma_{18}]=-i\Gamma_{23};\quad [\Gamma_{10},\Gamma_{19}]=\Gamma_{27};$

$[\Gamma_{10},\Gamma_{20}]=-i\Gamma_{27};\quad [\Gamma_{10},\Gamma_{29}]=2\Gamma_{10};\quad [\Gamma_{10},\Gamma_{30}]=\Gamma_{11};$

$[\Gamma_{10},\Gamma_{31}]=\Gamma_{15};\quad [\Gamma_{10},\Gamma_{32}]=\Gamma_{13};\quad [\Gamma_{10},\Gamma_{33}]=-2i\Gamma_{10};$

$[\Gamma_{10},\Gamma_{34}]=-i\Gamma_{11};\quad [\Gamma_{10},\Gamma_{35}]=-i\Gamma_{13};\quad [\Gamma_{10},\Gamma_{36}]=-i\Gamma_{15};$

$[\Gamma_{11},\Gamma_{17}]=\Gamma_{24}-i\Gamma_{23};\quad [\Gamma_{11},\Gamma_{18}]=-\Gamma_{23}-i\Gamma_{24};\quad [\Gamma_{11},\Gamma_{19}]=\Gamma_{28}-i\Gamma_{27};$

$[\Gamma_{11},\Gamma_{20}]=-i\Gamma_{28}-\Gamma_{27};\quad [\Gamma_{11},\Gamma_{29}]=\Gamma_{11}-2i\Gamma_{10};\quad [\Gamma_{11},\Gamma_{30}]=2\Gamma_{12}-i\Gamma_{11};$

$[\Gamma_{11},\Gamma_{31}]=\Gamma_{16}-i\Gamma_{15};\quad [\Gamma_{11},\Gamma_{32}]=\Gamma_{14}-i\Gamma_{13};\quad [\Gamma_{11},\Gamma_{33}]=-2\Gamma_{10}-i\Gamma_{11};$

$[\Gamma_{11},\Gamma_{34}]=-2i\Gamma_{12}-i\Gamma_{11};\quad [\Gamma_{11},\Gamma_{35}]=-i\Gamma_{14}-\Gamma_{13};\quad [\Gamma_{11},\Gamma_{36}]=-i\Gamma_{16}-\Gamma_{15};$

$[\Gamma_{12},\Gamma_{17}]=-i\Gamma_{24};\quad [\Gamma_{12},\Gamma_{18}]=-\Gamma_{24};\quad [\Gamma_{12},\Gamma_{19}]=-i\Gamma_{28};$

$[\Gamma_{12},\Gamma_{20}]=-\Gamma_{28};\quad [\Gamma_{12},\Gamma_{29}]=-i\Gamma_{11};\quad [\Gamma_{12},\Gamma_{30}]=-2i\Gamma_{12};$

$[\Gamma_{12},\Gamma_{31}]=-i\Gamma_{16};\quad [\Gamma_{12},\Gamma_{32}]=-i\Gamma_{14};\quad [\Gamma_{12},\Gamma_{33}]=-\Gamma_{11};$

$[\Gamma_{12},\Gamma_{34}]=-2\Gamma_{12};\quad [\Gamma_{12},\Gamma_{35}]=-\Gamma_{14};\quad [\Gamma_{12},\Gamma_{36}]=-\Gamma_{16};$

$[\Gamma_{13},\Gamma_{17}]=\Gamma_{29}+\Gamma_{22};\quad [\Gamma_{13},\Gamma_{18}]=\Gamma_{33}-i\Gamma_{22};\quad [\Gamma_{13},\Gamma_{19}]=\Gamma_{26}-i\Gamma_{29};$

$[\Gamma_{13},\Gamma_{20}]=-i\Gamma_{33}-i\Gamma_{26};\quad [\Gamma_{13},\Gamma_{21}]=\Gamma_{15};\quad [\Gamma_{13},\Gamma_{22}]=\Gamma_{13};$

$[\Gamma_{13},\Gamma_{23}]=2\Gamma_{10};\quad [\Gamma_{13},\Gamma_{24}]=\Gamma_{11};\quad [\Gamma_{13},\Gamma_{25}]=-i\Gamma_{15};$

$[\Gamma_{13},\Gamma_{26}]=-i\Gamma_{13};\quad [\Gamma_{13},\Gamma_{27}]=-2i\Gamma_{10};\quad [\Gamma_{13},\Gamma_{28}]=-i\Gamma_9;$

$[\Gamma_{13},\Gamma_{29}]=\Gamma_{13};\quad [\Gamma_{13},\Gamma_{30}]=\Gamma_{14};\quad [\Gamma_{13},\Gamma_{31}]=\Gamma_9;$

$[\Gamma_{13},\Gamma_{32}]=2\Gamma_7;\quad [\Gamma_{13},\Gamma_{33}]=-i\Gamma_{13};\quad [\Gamma_{13},\Gamma_{34}]=-i\Gamma_{14};\ [\Gamma_{13},\Gamma_{35}]=-2i\Gamma_7;\quad [\Gamma_{13},\Gamma_{36}]=-i\Gamma_9;$

$[\Gamma_{14},\Gamma_{17}]=\Gamma_{30}-i\Gamma_{16};\quad [\Gamma_{14},\Gamma_{18}]=\Gamma_{34}-\Gamma_{22};\quad [\Gamma_{14},\Gamma_{19}]=-i\Gamma_{30}-i\Gamma_{26};$

$[\Gamma_{14},\Gamma_{20}]=-\Gamma_{26}-i\Gamma_{34};\quad [\Gamma_{14},\Gamma_{21}]=\Gamma_{16};\quad [\Gamma_{14},\Gamma_{22}]=\Gamma_{14};$

$[\Gamma_{14},\Gamma_{23}]=\Gamma_{11};\quad [\Gamma_{14},\Gamma_{24}]=2\Gamma_{12};\quad [\Gamma_{14},\Gamma_{25}]=-i\Gamma_{16};$

$[\Gamma_{14},\Gamma_{26}]=-i\Gamma_{14};\quad [\Gamma_{14},\Gamma_{27}]=-i\Gamma_{11};\quad [\Gamma_{14},\Gamma_{28}]=-2i\Gamma_{12};$

$[\Gamma_{14},\Gamma_{29}]=-i\Gamma_{13};\quad [\Gamma_{14},\Gamma_{30}]=-i\Gamma_{14};\quad [\Gamma_{14},\Gamma_{31}]=-i\Gamma_{19};$

$[\Gamma_{14},\Gamma_{32}]=-2i\Gamma_7;\quad [\Gamma_{14},\Gamma_{33}]=-\Gamma_{13};\quad [\Gamma_{14},\Gamma_{34}]=-\Gamma_{14};\ [\Gamma_{14},\Gamma_{35}]=-2\Gamma_7;\quad [\Gamma_{14},\Gamma_{36}]=-\Gamma_9;$

$[\Gamma_{15},\Gamma_{17}]=-i\Gamma_{29}+\Gamma_{21};\quad [\Gamma_{15},\Gamma_{18}]=-i\Gamma_{33}-i\Gamma_{21};\quad [\Gamma_{15},\Gamma_{19}]=-\Gamma_{29}+\Gamma_{25};$

$[\Gamma_{15},\Gamma_{20}]=-\Gamma_{33}-i\Gamma_{25};\quad [\Gamma_{15},\Gamma_{21}]=-i\Gamma_{15};\quad [\Gamma_{15},\Gamma_{22}]=-i\Gamma_{13};$

$[\Gamma_{15},\Gamma_{23}]=-2i\Gamma_{10};\quad [\Gamma_{15},\Gamma_{24}]=-i\Gamma_{11};\quad [\Gamma_{15},\Gamma_{25}]=-\Gamma_{15};$

$$
\begin{aligned}
&[\Gamma_{15},\Gamma_{26}]=-\Gamma_{13};\quad [\Gamma_{15},\Gamma_{27}]=-2\Gamma_{10};\quad [\Gamma_{15},\Gamma_{28}]=-\Gamma_{11};\\
&[\Gamma_{15},\Gamma_{29}]=\Gamma_{15};\quad [\Gamma_{15},\Gamma_{30}]=\Gamma_{16};\quad [\Gamma_{15},\Gamma_{31}]=2\Gamma_{8};\\
&[\Gamma_{15},\Gamma_{32}]=\Gamma_{9};\quad [\Gamma_{15},\Gamma_{33}]=-i\Gamma_{15};\quad [\Gamma_{15},\Gamma_{34}]=-i\Gamma_{16};\ [\Gamma_{15},\Gamma_{35}]=-i\Gamma_{9};\quad [\Gamma_{15},\Gamma_{36}]=-2i\Gamma_{8};\\
&[\Gamma_{16},\Gamma_{17}]=-i\Gamma_{30}-i\Gamma_{21};\quad [\Gamma_{16},\Gamma_{18}]=-i\Gamma_{34}-\Gamma_{21};\quad [\Gamma_{16},\Gamma_{19}]=-\Gamma_{30}-i\Gamma_{25};\\
&[\Gamma_{16},\Gamma_{20}]=-i\Gamma_{16};\quad [\Gamma_{16},\Gamma_{21}]=-i\Gamma_{16};\quad [\Gamma_{16},\Gamma_{22}]=-i\Gamma_{14};\\
&[\Gamma_{16},\Gamma_{23}]=-i\Gamma_{11};\quad [\Gamma_{16},\Gamma_{24}]=-2i\Gamma_{12};\quad [\Gamma_{16},\Gamma_{25}]=-\Gamma_{16};\\
&[\Gamma_{16},\Gamma_{26}]=-\Gamma_{14};\quad [\Gamma_{16},\Gamma_{27}]=-\Gamma_{11};\quad [\Gamma_{16},\Gamma_{28}]=-2\Gamma_{12};\\
&[\Gamma_{16},\Gamma_{29}]=-i\Gamma_{15};\quad [\Gamma_{16},\Gamma_{30}]=-i\Gamma_{16};\quad [\Gamma_{16},\Gamma_{31}]=-2i\Gamma_{8};\\
&[\Gamma_{16},\Gamma_{32}]=-i\Gamma_{9};\quad [\Gamma_{16},\Gamma_{33}]=-\Gamma_{15};\quad [\Gamma_{16},\Gamma_{34}]=\Gamma_{16};\\
&[\Gamma_{16},\Gamma_{35}]=-\Gamma_{9};\quad [\Gamma_{16},\Gamma_{36}]=-2\Gamma_{2};\quad [\Gamma_{17},\Gamma_{21}]=i\Gamma_{17};\quad [\Gamma_{17},\Gamma_{22}]=-\Gamma_{17};\\
&[\Gamma_{17},\Gamma_{23}]=-2\Gamma_{1};\quad [\Gamma_{17},\Gamma_{24}]=2i\Gamma_{1};\quad [\Gamma_{17},\Gamma_{25}]=i\Gamma_{19};\\
&[\Gamma_{17},\Gamma_{26}]=-\Gamma_{19};\quad [\Gamma_{17},\Gamma_{27}]=-\Gamma_{3};\quad [\Gamma_{17},\Gamma_{28}]=i\Gamma_{3};\\
&[\Gamma_{17},\Gamma_{29}]=-\Gamma_{17};\quad [\Gamma_{17},\Gamma_{30}]=i\Gamma_{17};\quad [\Gamma_{17},\Gamma_{31}]=2i\Gamma_{4};\\
&[\Gamma_{17},\Gamma_{32}]=-2\Gamma_{4};\quad [\Gamma_{17},\Gamma_{33}]=-\Gamma_{18};\quad [\Gamma_{17},\Gamma_{34}]=-i\Gamma_{18};\\
&[\Gamma_{17},\Gamma_{35}]=-\Gamma_{6};\quad [\Gamma_{17},\Gamma_{36}]=i\Gamma_{6};\quad [\Gamma_{18},\Gamma_{21}]=i\Gamma_{18};\quad [\Gamma_{18},\Gamma_{22}]=-\Gamma_{18};\\
&[\Gamma_{18},\Gamma_{23}]=2i\Gamma_{1};\quad [\Gamma_{18},\Gamma_{24}]=2\Gamma_{1};\quad [\Gamma_{18},\Gamma_{25}]=i\Gamma_{20};\\
&[\Gamma_{18},\Gamma_{26}]=-\Gamma_{20};\quad [\Gamma_{18},\Gamma_{27}]=\Gamma_{3};\quad [\Gamma_{18},\Gamma_{28}]=\Gamma_{3};\\
&[\Gamma_{18},\Gamma_{29}]=i\Gamma_{17};\quad [\Gamma_{18},\Gamma_{30}]=\Gamma_{17};\quad [\Gamma_{18},\Gamma_{31}]=i\Gamma_{6};\\
&[\Gamma_{18},\Gamma_{32}]=-\Gamma_{6};\quad [\Gamma_{18},\Gamma_{33}]=i\Gamma_{18};\quad [\Gamma_{18},\Gamma_{34}]=-\Gamma_{18};\\
&[\Gamma_{18},\Gamma_{35}]=-2\Gamma_{5};\quad [\Gamma_{18},\Gamma_{36}]=2i\Gamma_{5};\quad [\Gamma_{19},\Gamma_{21}]=\Gamma_{17};\quad [\Gamma_{19},\Gamma_{22}]=i\Gamma_{17};\\
&[\Gamma_{19},\Gamma_{23}]=-\Gamma_{3};\quad [\Gamma_{19},\Gamma_{24}]=i\Gamma_{3};\quad [\Gamma_{19},\Gamma_{25}]=\Gamma_{19};\\
&[\Gamma_{19},\Gamma_{26}]=i\Gamma_{19};\quad [\Gamma_{19},\Gamma_{27}]=-2\Gamma_{2};\quad [\Gamma_{19},\Gamma_{28}]=2i\Gamma_{2};\\
&[\Gamma_{19},\Gamma_{29}]=-i\Gamma_{19};\quad [\Gamma_{19},\Gamma_{30}]=i\Gamma_{19};\quad [\Gamma_{19},\Gamma_{31}]=2\Gamma_{4};\\
&[\Gamma_{19},\Gamma_{32}]=2i\Gamma_{4};\quad [\Gamma_{19},\Gamma_{33}]=-\Gamma_{20};\quad [\Gamma_{19},\Gamma_{34}]=i\Gamma_{20};\\
&[\Gamma_{19},\Gamma_{35}]=i\Gamma_{6};\quad [\Gamma_{19},\Gamma_{36}]=\Gamma_{6};\quad [\Gamma_{20},\Gamma_{21}]=\Gamma_{18};\quad [\Gamma_{20},\Gamma_{22}]=i\Gamma_{18};\\
&[\Gamma_{20},\Gamma_{23}]=i\Gamma_{3};\quad [\Gamma_{20},\Gamma_{24}]=\Gamma_{3};\quad [\Gamma_{20},\Gamma_{25}]=\Gamma_{20};\\
&[\Gamma_{20},\Gamma_{26}]=i\Gamma_{20};\quad [\Gamma_{20},\Gamma_{27}]=2i\Gamma_{2};\quad [\Gamma_{20},\Gamma_{28}]=2\Gamma_{2};\\
&[\Gamma_{20},\Gamma_{29}]=i\Gamma_{19};\quad [\Gamma_{20},\Gamma_{30}]=\Gamma_{19};\quad [\Gamma_{20},\Gamma_{31}]=\Gamma_{6};\\
&[\Gamma_{20},\Gamma_{32}]=i\Gamma_{6};\quad [\Gamma_{20},\Gamma_{33}]=i\Gamma_{20};\quad [\Gamma_{20},\Gamma_{34}]=\Gamma_{20};\ [\Gamma_{20},\Gamma_{35}]=2i\Gamma_{5};\quad [\Gamma_{20},\Gamma_{36}]=2\Gamma_{5}.
\end{aligned}
\tag{3.170}
$$

Therefore, their use in eq.(3.66) yields the following set of PDEs in λ's:

$$\dot{\lambda}_1=4\alpha_1(i\lambda_{21}-\lambda_{22})-4\alpha_3(\lambda_{23}-i\lambda_{24}), \tag{3.171}$$

$$\dot{\lambda}_2=-4\alpha_1(\lambda_{25}+i\lambda_{26})-4\alpha_3(i\lambda_{27}+\lambda_{28}), \tag{3.172}$$

$$\dot{\lambda}_3=-2\alpha_1(\lambda_{21}+i\lambda_{22}+i\lambda_{25}-\lambda_{26})-2\alpha_3(i\lambda_{23}+\lambda_{24}+\lambda_{27}-i\lambda_{28}), \tag{3.173}$$

$$\dot{\lambda}_4=-4\alpha_2(\lambda_{29}+i\lambda_{30})+4\alpha_3(i\lambda_{31}-\lambda_{32}), \tag{3.174}$$

$$\dot{\lambda}_5=-4\alpha_2(\lambda_{34}+i\lambda_{33})-4\alpha_3(\lambda_{36}+i\lambda_{35}), \tag{3.175}$$

$$\dot{\lambda}_6=-2\alpha_2(i\lambda_{29}+\lambda_{30}-i\lambda_{34}+\lambda_{33})-2\alpha_3(\lambda_{31}+i\lambda_{32}+\lambda_{35}-i\lambda_{36}), \tag{3.176}$$

$$\dot{\lambda}_7=-4\beta_1(-\lambda_{22}+i\lambda_{26})+4\beta_3(\lambda_{32}-i\lambda_{35}), \tag{3.177}$$

$$\dot{\lambda}_8 = 4\beta_1(i\lambda_{21} + \lambda_{25}) + 4\beta_3(\lambda_{36} + i\lambda_{31}), \tag{3.178}$$

$$\dot{\lambda}_9 = 2\beta_1(\lambda_{21} + i\lambda_{22} - i\lambda_{25} + \lambda_{26}) + 2\beta_3(\lambda_{31} + i\lambda_{32} + \lambda_{35} - i\lambda_{36}), \tag{3.179}$$

$$\dot{\lambda}_{10} = 4\beta_3(\lambda_{23} - i\lambda_{27}) - 4\beta_2(\lambda_{29} + \lambda_{33}), \tag{3.180}$$

$$\dot{\lambda}_{11} = 2\beta_3(i\lambda_{23} + \lambda_{24} + \lambda_{27} - i\lambda_{28}) - 2\beta_3(\lambda_{29} + i\lambda_{30} + \lambda_{34} - i\lambda_{33}), \tag{3.181}$$

$$\dot{\lambda}_{12} = 4\beta_3(i\lambda_{24} + \lambda_{28}) - 4\beta_2(\lambda_{30} + \lambda_{34}), \tag{3.182}$$

$$\dot{\lambda}_{13} = 2\beta_3(\lambda_{22} - i\lambda_{33} - i\lambda_{26} + \lambda_{29}) + 2\beta_1(i\lambda_{23} - i\lambda_{27}) + 2\beta_2(\lambda_{32} - i\lambda_{35}), \tag{3.183}$$

$$\dot{\lambda}_{14} = 2\beta_3(\lambda_{26} + \lambda_{30} - i\lambda_{34} + i\lambda_{22}) + 2\beta_1(\lambda_{24} - i\lambda_{28}) + \beta_2(-\lambda_{32} + i\lambda_{35}), \tag{3.184}$$

$$\dot{\lambda}_{15} = 2\beta_3(\lambda_{21} - i\lambda_{25} + \lambda_{33} + i\lambda_{29}) + 2\beta_1(i\lambda_{23} + \lambda_{27}) + \beta_2(\lambda_{31} - \lambda_{36}), \tag{3.185}$$

$$\dot{\lambda}_{16} = 2\beta_2(i\lambda_{21} + \lambda_{25} + i\lambda_{30} + i\lambda_{34}) + 2\beta_1(i\lambda_{24} + \lambda_{28}) - \beta_2(\lambda_{31} + i\lambda_{36}), \tag{3.186}$$

$$\dot{\lambda}_{17} = 2\alpha_3(i\lambda_{21} - \lambda_{22} - \lambda_{29} + i\lambda_{30}) + 2\alpha_1(i\lambda_{31} - \lambda_{32}) - 2\alpha_2(i\lambda_{23} + \lambda_{24}), \tag{3.187}$$

$$\dot{\lambda}_{18} = -2\alpha_3(\lambda_{21} + i\lambda_{22} + i\lambda_{34} + \lambda_{33}) + 2\alpha_1(-\lambda_{31} + i\lambda_{32}) + \alpha_2(-\lambda_{27} + \lambda_{28}), \tag{3.188}$$

$$\dot{\lambda}_{19} = 2\alpha_3(i\lambda_{25} - \lambda_{26} - i\lambda_{29} - \lambda_{30}) - 2\alpha_1(\lambda_{31} + i\lambda_{32}) + 2\alpha_2(-\lambda_{27} + i\lambda_{28}), \tag{3.189}$$

$$\dot{\lambda}_{20} = -2\alpha_3(\lambda_{25} + i\lambda_{26} + i\lambda_{29} + \lambda_{30}) - 2\alpha_1(\lambda_{31} + i\lambda_{32}) + 2\alpha_2(-i\lambda_{27} - \lambda_{28}, \tag{3.190}$$

$$\dot{\lambda}_{21} = 2\beta_1(i\lambda_1 + \lambda_3) + 2\alpha_1(-i\lambda_9 + i\lambda_8) + 2\alpha_3(-i\lambda_{16} - \lambda_{15}) + 2\beta_3(i\lambda_{17} + \lambda_{18}), \tag{3.191}$$

$$\dot{\lambda}_{22} = 2\beta_1(\lambda_1 - i\lambda_3) + 2\alpha_1(i\lambda_9 - \lambda_7) + 2\alpha_3(i\lambda_{14} - \lambda_{13}) + 2\beta_3(\lambda_{17} - i\lambda_{18}), \tag{3.192}$$

$$\dot{\lambda}_{23} = 2\beta_2(i\lambda_{17} - i\lambda_{18}) + 2\alpha_1(-\lambda_{13} + i\lambda_{15}) + 2\beta_3(-i\lambda_3 + \lambda_1) + 2\alpha_3(i\lambda_{11} - \lambda_{10}), \tag{3.193}$$

$$\dot{\lambda}_{24} = 2\beta_3(i\lambda_1 + \lambda_3) + 2\alpha_3(-\lambda_{11} + i\lambda_{12}) + 2\alpha_1(-i\lambda_{16} - \lambda_{14}) + 2\beta_2(i\lambda_{18} - \lambda_{17}), \tag{3.194}$$

$$\dot{\lambda}_{25} = 2\beta_1(-i\lambda_2 - \lambda_3) - 2\alpha_1(i\lambda_9 + \lambda_8) + 2\alpha_3(-\lambda_{16} - i\lambda_{15}) + 2\beta_3(i\lambda_{19} + \lambda_{20}), \tag{3.195}$$

$$\dot{\lambda}_{26} = 2\beta_1(-i\lambda_2 + \lambda_3) + 2\alpha_1(-i\lambda_7 - \lambda_9) + 2\alpha_3(-i\lambda_{13} - \lambda_{14}) + 2\beta_3(\lambda_{19} - i\lambda_{20}), \tag{3.196}$$

$$\dot{\lambda}_{27} = 2\beta_3(-i\lambda_2 + \lambda_3) + 2\alpha_3(-i\lambda_{10} - \lambda_{11}) - 2\alpha_1(i\lambda_{13} + \lambda_{15}) + 2\beta_2(\lambda_{19} + i\lambda_{20}), \tag{3.197}$$

$$\dot{\lambda}_{28} = 2\beta_3(\lambda_2 + i\lambda_3) + 2\alpha_3(-i\lambda_{11} - i\lambda_{12}) - 2\alpha_1(i\lambda_{14} - \lambda_{16}) + 2\beta_2(-\lambda_{19} + i\lambda_{20}), \tag{3.198}$$

$$\dot{\lambda}_{29} = 2\beta_2(\lambda_4 + \lambda_6) + 2\alpha_2(-\lambda_{10} + i\lambda_{11}) + 2\alpha_3(i\lambda_{15} - \lambda_{13}) + 2\beta_3(\lambda_{17} - i\lambda_{19}), \tag{3.199}$$

$$\dot{\lambda}_{30} = -2\beta_3(\lambda_4 + i\lambda_6) + 2\alpha_2(-\lambda_{11} - i\lambda_{12}) + 2\alpha_3(-i\lambda_{16} - i\lambda_{14}) + 2\beta_3(i\lambda_{17} + \lambda_{19}), \tag{3.200}$$

$$\dot{\lambda}_{31} = 2\beta_3(i\lambda_4 + \lambda_6) + 2\alpha_2(i\lambda_8 + \lambda_9) + 2\alpha_2(-i\lambda_{16} - \lambda_{15}) + 2\beta_1(i\lambda_{17} + \lambda_{19}), \tag{3.201}$$

$$\dot{\lambda}_{32} = 2\beta_3(\lambda_4 - i\lambda_6) + 2\alpha_3(\lambda_7 + i\lambda_9) + 2\alpha_2(i\lambda_{14} - \lambda_{13}) + 2\beta_1(\lambda_{17} - i\lambda_{19}), \tag{3.202}$$

$$\dot{\lambda}_{33} = 2\beta_2(\lambda_5 + \lambda_6) + 2\alpha_2(-\lambda_{10} - i\lambda_{11}) + 2\alpha_3(-i\lambda_{13} - \lambda_{15}) + 2\beta_3(\lambda_{18} - i\lambda_{20}), \tag{3.203}$$

$$\dot{\lambda}_{34} = 2\beta_2(\lambda_5 + \lambda_6) + 2\alpha_2(-i\lambda_{11} + \lambda_{12}) + 2\alpha_3(-\lambda_{14} - i\lambda_{16}) + 2\beta_3(i\lambda_{18} + \lambda_{20}), \tag{3.204}$$

$$\dot{\lambda}_{35} = 2\beta_1(-\lambda_{18} - \lambda_{20}) + 2\alpha_2(-i\lambda_{13} - \lambda_{14}) + 2\beta_3(-i\lambda_5 + \lambda_6) + 2\alpha_3(i\lambda_7 - \lambda_9), \tag{3.205}$$

$$\dot{\lambda}_{36} = 2\beta_3(\lambda_5 + i\lambda_6) - 2\alpha_3(i\lambda_9 + \lambda_8) + 2\alpha_2(-\lambda_{16} + i\lambda_{15}) + 2\beta_1(i\lambda_{18} + \lambda_{20}), \tag{3.206}$$

In fact, to solve these 36 coupled PDEs for complex λ's is very difficult. Thus, here we make certain choices for λ's which facilitate to find solutions of above equations. From above eqs., we get $2\dot{\lambda}_3 = i\dot{\lambda}_1 - i\dot{\lambda}_2$, and if we consider $\lambda_3 = c_3$ (a constant), and consider relation $\lambda_1 = \lambda_2 = \eta_1(t)$, which immediately gives

$$\lambda_1 = \eta_1(t) + c_1; \qquad \lambda_2 = \eta_1(t) + c_2. \tag{3.207}$$

again we have $2\dot{\lambda}_6 = i\dot{\lambda}_4 - i\dot{\lambda}_5$, and if we consider $\lambda_6 = c_6$ (a constant), and consider relation $\lambda_4 = \lambda_5 = \eta_2(t)$, which immediately gives

$$\lambda_4 = \eta_2(t) + c_4; \qquad \lambda_5 = \eta_2(t) + c_5. \tag{3.208}$$

From above eqs, we get $2\dot{\lambda}_9 = i\dot{\lambda}_7 - i\dot{\lambda}_8$, and if we consider $\lambda_9 = c_9$ (a constant),and consider relation $\lambda_7 = \lambda_8 = \eta_3(t)$, which immediately gives

$$\lambda_7 = \eta_3(t) + c_7; \qquad \lambda_8 = \eta_3(t) + c_8. \tag{3.209}$$

Similarly , we have $2\dot{\lambda}_{11} = i\dot{\lambda}_{10} - i\dot{\lambda}_{12}$, and if we consider $\lambda_{11} = c_{11}$ (a constant), and consider relation $\lambda_{12} = \lambda_{10} = \eta_4(t)$, which immediately gives

$$\lambda_{12} = \eta_4(t) + c_{12}; \qquad \lambda_{10} = \eta_4(t) + c_{10}. \tag{3.210}$$

Now, in order to find solutions for λ_{13}, λ_{14}, λ_{15} and λ_{16} we have to make simplification for complications of above set of 24 equations. (i.e. $\alpha_1 = \alpha_2 = \alpha_3$, and $\beta_1 = \beta_2 = \beta_3$.) One can have $2\dot{\lambda}_{13} = \dot{\lambda}_7 + i\dot{\lambda}_{10}$, and if we consider $\lambda_{13} = c_{13}$ (a constant), further from above relation (with $\lambda_{10} = \eta_4(t)+c_{10};\ \ \lambda_7 = \eta_3(t)+c_7$) gives

$$\lambda_{13} = \eta(t) + c_{13}. \tag{3.211}$$

where $\eta(t) = \frac{1}{2i}[\eta_4(t) + \eta_3(t)]$; is an another function of time and $c_{13} = c_7 + c_{10}$; a constant Similarly, we get $2i\dot{\lambda}_{15} = \dot{\lambda}_{10} + i\dot{\lambda}_8$, and if we consider $\lambda_{15} = c_{15}$ (a constant), further from above relation (with $\lambda_{10} = \eta_4(t) + c_{10}; \lambda_8 = \eta_3(t) + c_8$), gives

$$\lambda_{15} = \eta(t) + c_{15}. \tag{3.212}$$

where $\eta(t) = \frac{1}{2i}[\eta_4(t) + \eta_3(t)]$ is an another function of time and $c_{15} = c_{10} + c_8$; a constant.
Again, in order to find solutions for λ_{16}, from eqs.(3.178), (3.182) and (3.186), we get $2i\dot{\lambda}_{16} = \dot{\lambda}_8+\dot{\lambda}_{12}$, and if we consider $\lambda_{16} = c_{16}$ (a constant), further from above relation (with $\lambda_8 = \eta_3(t)+c_8;\ \ \lambda_{12} = \eta_4(t)+c_{12}$) gives

$$\lambda_{16} = \eta(t) + c_{16}. \tag{3.213}$$

where $\eta(t) = \frac{1}{2i}[\eta_4(t) + \eta_3(t)]$; and $c_{16} = c_8 + c_{12}$; a constant. From eqs.(3.177), (3.182) and (3.184), we get $2i\dot{\lambda}_{14} = \dot{\lambda}_7 + \dot{\lambda}_{12}$, and if we consider $\lambda_{14} = c_{14}$ (a constant), further from above relation (with $\lambda_7 = \eta_3(t) + c_7; \quad \lambda_{12} = \eta_4(t) + c_{12}$) gives

$$\lambda_{14} = \eta(t) + c_{14}. \tag{3.214}$$

where $\eta(t) = \frac{1}{2i}[\eta_4(t) + \eta_3(t)]$; and $c_{16} = c_7 + c_{12}$; a constant.
Now, in order to find solutions for λ_{17} to λ_{20}, Refer from eqs.(3.171), (3.174) and (3.187), we get $2\dot{\lambda}_{17} = \dot{\lambda}_1 + \dot{\lambda}_4$, and, further from the relation, with

$$\lambda_1 = \eta_1(t) + c_1; \qquad \lambda_4 = \eta_2(t) + c_4.$$

gives

$$\lambda_{17} = \phi(t) + c_{17} \tag{3.215}$$

where $\phi(t) = \frac{1}{2}\int[\dot{\eta}_1(t) + \dot{\eta}_2(t)]dt$ is an another function of t and ($c_{17} = c_1 + c_4$, a constant)
From eqs.(3.171), (3.175) and (3.188), we get $2\dot{\lambda}_{18} = i\dot{\lambda}_1 - i\dot{\lambda}_5$, and, further from above relation, with

$$\lambda_1 = \eta_1(t) + c_1; \qquad \lambda_5 = \eta_2(t) + c_5.$$

gives

$$\lambda_{18} = \varphi(t) + c_{18} \tag{3.216}$$

where $\varphi(t) = \frac{1}{2}\int[i\dot{\eta}_1(t) - i\dot{\eta}_2(t)]dt$; and ($c_{18} = c_1 + c_5$, a constant).

Again, in order to find solutions for λ_{19}, from eqs.(3.172), (3.174) and (3.189), we get $2\dot{\lambda}_{19} = i\dot{\lambda}_2 - i\dot{\lambda}_4$, and, further from above relation, with

$$\lambda_2 = \eta_1(t) + c_2; \qquad \lambda_4 = \eta_2(t) + c_4.$$

gives

$$\lambda_{19} = \chi(t) + c_{19}. \tag{3.217}$$

where $\chi(t) = \frac{1}{2}\int[i\dot{\eta}_1(t) - i\dot{\eta}_2(t);]dt$ and $c_{19} = c_2 + c_4$, a constant.

From eqs.(3.172) , (3.175) and (3.190), we get $2\dot{\lambda}_{20} = \dot{\lambda}_2 + \dot{\lambda}_5$, and, further from above relation, with

$$\lambda_2 = \eta_1(t) + c_1; \qquad \lambda_5 = \eta_2(t) + c_5.$$

gives

$$\lambda_{20} = \psi(t) + c_{20}. \tag{3.218}$$

where $\psi(t) = \frac{1}{2}\int[\dot{\eta}_1(t) + \dot{\eta}_2(t)]dt$ is an another function of t and ($c_{20} = c_2 + c_5$, a constant)

Now to obtain solutions for (λ_{21} to λ_{28}) respectively as, from eqs.(3.191 to 3.198), we obtain following equations

$$i\dot{\lambda}_{21} + \dot{\lambda}_{22} = 2(\alpha_1\lambda_7 - \alpha_1\lambda_8 - \alpha_3\lambda_{13} + i\alpha_3\lambda_{14} - i\alpha_3\lambda_{15} + \alpha_3\lambda_{16}) = 0. \tag{3.219}$$

$$\dot{\lambda}_{23} + i\dot{\lambda}_{24} = 2(-\alpha_3\lambda_{10} + \alpha_3\lambda_{12} - \alpha_1\lambda_{13} + i\alpha_1\lambda_{15} - i\alpha_1\lambda_{14} + \alpha_1\lambda_{16}) = 0. \tag{3.220}$$

$$\dot{\lambda}_{25} - i\dot{\lambda}_{26} = 2(\alpha_1\lambda_7 - \alpha_1\lambda_8 - \alpha_3\lambda_{13} + i\alpha_3\lambda_{14} - i\alpha_3\lambda_{15} + \alpha_3\lambda_{16}) = 0. \tag{3.221}$$

$$\dot{\lambda}_{27} + i\dot{\lambda}_{28} = 2i(-\alpha_3\lambda_{10} + \alpha_3\lambda_{12} - \alpha_1\lambda_{13} + i\alpha_1\lambda_{15} - i\alpha_1\lambda_{14} + \alpha_1\lambda_{16}) = 0. \tag{3.222}$$

since

$$\lambda_7 = \lambda_8, \lambda_{10} = \lambda_{12}; \text{and} \lambda_{13} = \lambda_{14} = \lambda_{15} = \lambda_{16} = \eta(t).$$

or if we set

$$\dot{\lambda}_{21} = i\dot{\lambda}_{22} = \dot{\xi}(t); \qquad \dot{\lambda}_{23} = -i\dot{\lambda}_{24} = \dot{\theta}(t)$$

$$\dot{\lambda}_{25} = i\dot{\lambda}_{26} = \dot{\delta}(t); \quad \dot{\lambda}_{27} = -i\dot{\lambda}_{28} = \dot{\zeta}(t) \tag{3.223}$$

which immediately gives

$$\lambda_{21} = \xi(t) + c_{21}, \qquad \lambda_{22} = -i\xi(t) + c_{22}; \qquad \lambda_{23} = \theta(t) + c_{22}, \qquad \lambda_{24} = -i\theta(t) + c_{29}$$

$$\lambda_{25} = \delta(t)c_{25}, \qquad \lambda_{26} = -i\delta(t) + c_{30}; \qquad \lambda_{27} = \zeta(t) + c_{28}, \qquad \lambda_{28} = i\zeta(t) + c_{32} \tag{3.224}$$

Similarly, to obtain solutions for (λ_{29} to λ_{36}) respectively as, refer from eqs.(3.199 to 3.206), we obtain following equations

$$\dot{\lambda}_{31} - i\dot{\lambda}_{32} = -2(-i\alpha_3\lambda_7 + i\alpha_3\lambda_8 + i\alpha_2\lambda_{13} + \alpha_2\lambda_{14} - \alpha_2\lambda_{15} - i\alpha_2\lambda_{16}) = 0. \tag{3.225}$$

$$\dot{\lambda}_{29} + i\dot{\lambda}_{30} = 2(-\alpha_2\lambda_{10} + \alpha_2\lambda_{12} - \alpha_3\lambda_{13} + i\alpha_3\lambda_{15} - i\alpha_3\lambda_{14} + \alpha_3\lambda_{16}) = 0. \tag{3.226}$$

$$\dot{\lambda}_{33} + i\dot{\lambda}_{34} = 2i(-\alpha_2\lambda_{10} + \alpha_2\lambda_{12} - \alpha_3\lambda_{13} + i\alpha_3\lambda_{15} - i\alpha_3\lambda_{14} + \alpha_3\lambda_{16}) = 0. \tag{3.227}$$

$$\dot{\lambda}_{35} + i\dot{\lambda}_{36} = -2(-i\alpha_3\lambda_7 + i\alpha_3\lambda_8 + i\alpha_2\lambda_{13} + \alpha_2\lambda_{14} - \alpha_2\lambda_{15} - i\alpha_2\lambda_{16}) = 0. \tag{3.228}$$

since

$$\lambda_7 = \lambda_8, \lambda_{10} = \lambda_{12}; \text{and} \lambda_{13} = \lambda_{14} = \lambda_{15} = \lambda_{16} = \eta(t).$$

or if we set

$$\dot{\lambda}_{29} = -i\dot{\lambda}_{30} = \dot{\gamma}(t); \quad \dot{\lambda}_{31} = i\dot{\lambda}_{32} = \dot{\mu}(t) \quad \dot{\lambda}_{33} = -i\dot{\lambda}_{34} = \dot{\rho}(t); \quad \dot{\lambda}_{35} = -i\dot{\lambda}_{36} = \dot{\sigma}(t). \tag{3.229}$$

which immediately gives

$$\lambda_{29} = \gamma(t) + c_{19}, \qquad \lambda_{30} = i\gamma(t) + c_{31}; \qquad \lambda_{31} = \mu(t) + c_{22}, \qquad \lambda_{32} = -i\mu(t) + c_{29}$$

$$\lambda_{33} = \rho(t)c_{25}, \qquad \lambda_{34} = i\rho(t) + c_{30}; \qquad \lambda_{35} = \sigma(t) + c_{28}, \qquad \lambda_{36} = i\sigma(t) + c_{32} \tag{3.230}$$

We have solved eqs.[(3.171) to (3.206)] in terms of arbitrary functions η's, $\phi, \varphi, \chi, \psi, \xi, \theta, \delta, \zeta, \gamma, \mu, \rho$ and σ's and complex constants, c_i's, $(i = 1, ..., 36)$. If one put back these solutions for λ_i, $(i = 1, ..., 36)$ in eqs. [(3.171) to (3.206)], we obtain a number of constraint relations among c_i's, and η's, $\phi, \varphi, \chi, \psi, \xi, \theta, \delta, \zeta, \gamma, \mu, \rho$ and σ's, which limit the choices of these arbitrary complex quantities. If we set $c_i's = 0$, then with the equations determining arbitrary functions η's, $\phi, \varphi, \chi, \psi, \xi, \theta, \delta, \zeta, \gamma, \mu, \rho$ and σ's are written as

$$\dot{\theta} - i\dot{\zeta} = 0, \;\; \alpha_2(\dot{\gamma} - i\dot{\rho}) + \alpha_3(\dot{\mu} - i\dot{\sigma}) = 0; \quad \ddot{\eta}_3 - 4\beta(\dot{\xi} + \dot{\delta} - \dot{\sigma} + \dot{\mu}) = 0,$$

$$\dot{\mu} - \dot{\sigma} = 0;\ i\dot{\theta} + \dot{\zeta} = i(\dot{\gamma} + \dot{\rho});\quad \ddot{\eta}_4 - 4\beta(i\dot{\theta} + \dot{\zeta} + \dot{\gamma} + \dot{\rho}) = 0,$$

$$\ddot{\eta} - 2\beta(\dot{\xi} - \dot{\rho} + \dot{\delta} - i\dot{\gamma} + \dot{\theta} + \dot{\zeta} - \dot{\mu} + \dot{\sigma}) = 0;\quad \ddot{\eta} + 2i\beta(\dot{\xi} - \dot{\rho} + \dot{\delta} + i\dot{\gamma} + \dot{\theta} + \dot{\zeta} - \dot{\mu} - i\dot{\sigma}) = 0,$$

$$\ddot{\chi} + 4\alpha(\dot{\gamma} - i\dot{\zeta} + i\dot{\mu}) = 0,\ \ddot{\psi} + 4\alpha(\dot{\rho} + \dot{\zeta}) = 0,;\quad \ddot{\xi} - 2\beta(i\dot{\eta}_1 + i\dot{\phi} + \dot{\varphi}) + 2\alpha(i\dot{\eta}_3 - i\dot{\eta} + \dot{\eta}) = 0,$$

$$\ddot{\theta} - 2i\beta(\dot{\phi} - \dot{\varphi} + i\dot{\eta}_1) + 2i\alpha(i\dot{\eta}_1 + \dot{\eta} + \dot{\eta}_4) = 0;\quad \ddot{\delta} - 2\beta(i\dot{\chi} + \dot{\phi} + \dot{\eta}_1) + 2\alpha(\dot{\eta}_3 + i\dot{\eta} + \dot{\eta}) = 0,$$

$$\ddot{\zeta} + 2i\beta(i\dot{\psi} - i\dot{\eta}_2 + \dot{\chi}) - 2i\alpha(\dot{\eta} - i\dot{\eta} - i\dot{\eta}_4) = 0;\quad \ddot{\gamma} - 2i\beta(i\dot{\eta}_2 + \dot{\phi} - i\dot{\chi}) + 2i\alpha(i\dot{\eta}_4 - i\dot{\eta} + \dot{\eta}) = 0,$$

$$\ddot{\lambda} + 2i\beta(i\dot{\eta}_2 + i\dot{\phi} + \dot{\chi}) + 2i\alpha(i\dot{\eta}_3 + i\dot{\eta} - \dot{\eta}) = 0;\quad \ddot{\rho} - 2i\beta(i\dot{\eta}_2 - i\dot{\phi} + \dot{\varphi}) + 2i\alpha(i\dot{\eta}_4 + i\dot{\eta} + \dot{\eta}) = 0,$$

$$\ddot{\sigma} + 2i\beta(i\dot{\eta}_2 + i\dot{\phi} + \dot{\varphi}) + 2i\alpha(-i\dot{\eta}_3 + i\dot{\eta} + \dot{\eta}) = 0, \tag{3.231}$$

Therefore, after substituting the solutions of λ_i's in the eq.(3.66), the complex invariant for a two dimensional complex oscillator becomes

$$\begin{aligned} I = {}& \frac{1}{2}\eta_1(p_1^2 + x_3^2) + \frac{1}{2}\eta_2(p_2^2 + x_4^2) + \frac{1}{2}\eta_4(x_1^2 + p_3^2) + \frac{1}{2}\eta_3(x_2^2 + p_4^2) + \eta(x_1x_2 + p_3p_4 + x_2p_3 + x_1p_4) \\ & +\xi(p_1p_3 - ip_1x_1) + \theta(p_1x_2 - ip_1p_4) + \delta(p_3x_3 - ix_1x_3) + \zeta(ip_4x_3 + x_2x_3) + \gamma(ip_2p_4 + p_2x_2) \\ & +\mu(p_3p_2 - ix_1p_2) + \rho(ip_4x_4 + x_2x_4) + \sigma(ip_3x_4 + x_1x_4) \end{aligned} \tag{3.232}$$

which conforms to condition eq.(3.66) in view of the Poisson bracket eq.(3.67). As pointed out in section 1, firstly the existence of an invariant for a dynamical system is questionable. If the invariant exists, then its construction, in general, is a difficult task. Once it is constructed and becomes available then not only its physical interpretation(s) but also its viability with regard to a better theoretical understanding of a given phenomenon is often a problem. In spite of all this, the availability of a few or all invariants for a dynamical system definitely offers insight into the finer details as far as an understanding of the phenomenon is concerned. In this work a modest attempt has been made to obtain exact complex second constant of motion of a two dimensional complex coupled harmonic oscillator on an extended complex phase space. In coupled mechanical oscillator problems, the transfer of energy from one oscillator to the other is attributed mainly to the x_1x_2-coupling term in the Hamiltonian. In the description using equivalent versions (invariants) these features might manifest through some other terms may be through the momentum-dependent ones. These complex invariants could be helpful to get some better understanding of the dynamical systems.

3.3 Construction of complex invariants in three-dimensions

The method commonly used for TD systems in the literature is the direct method or rationalization method. Although this method has been extended to 2D systems earlier [16], however, it turns out that the degree of complexity for the 2D case further increases. Therefore, in the present work, we resort to the dynamical algebraic approach which has so far been applied successfully either to one-dimensional systems or to a restricted class of higher dimensional systems. To the best of our knowledge, this latter method has not been exploited thus far for the case of three-dimensional complex systems. However for three-dimensional real systems an invariant was derived by Kausal *et al* [19]. In what follows we briefly outline the essential steps of the dynamical algebraic method for the 3D case.

The method for three-dimensions

Consider a three-dimensional real phase space (x, y, z, p_x, p_y, p_z), which may be transformed into a complex space $(x_1, p_1, x_2, p_2, x_3, p_3, x_4, p_4, x_5, p_5, x_6, p_6)$, by defining position and momenta variables as

$$\begin{aligned} x &= x_1 + ip_4; \qquad y = x_2 + ip_5; \qquad z = x_3 + ip_5; \\ p_x &= p_1 + ix_4; \qquad p_y = p_2 + ix_5; \qquad p_z = p_3 + ix_6. \end{aligned} \tag{3.233}$$

the presence of variables $(x_4, x_5, x_6, p_4, p_5, p_6)$ in the above transformation eq.(3.233), are some sort of coordinate-momentum interaction of the dynamical system. The Hamiltonian $H(x, y, z, p_x, p_y, p_z)$ of a three-dimensional system in complex space can be expressed, it as $H = H_1 + iH_2$. Using eq.(3.233), as the complex Hamiltonian $H(x, p, t)$ turns out to be

$$H = \sum_n h_n(t)\Gamma_n(x_1, p_1, x_2, p_2, x_3, p_3, x_4, p_4, x_5, p_5, x_6, p_6), \tag{3.234}$$

where the set of functions $\{\Gamma_1, ..., \Gamma_n\}$ are not explicitly time dependent and $h_n(t)$ are complex coefficient functions of time. The Γ_n's in eq.(3.1) generate a closed dynamical algebra, implies

$$[\Gamma_n, \Gamma_m] = \sum_l C^l_{nm}\Gamma_l, \tag{3.235}$$

However for three dimensional real systems Poison bracket turns out to be

$$[\Gamma_n, \Gamma_m] = \frac{\partial\Gamma_n}{\partial x}\frac{\partial\Gamma_m}{\partial p_x} - \frac{\partial\Gamma_n}{\partial p_x}\frac{\partial\Gamma_m}{\partial x} + \frac{\partial\Gamma_n}{\partial y}\frac{\partial\Gamma_m}{\partial p_y} - \frac{\partial\Gamma_n}{\partial p_y}\frac{\partial\Gamma_m}{\partial y} + \frac{\partial\Gamma_n}{\partial z}\frac{\partial\Gamma_m}{\partial p_z} - \frac{\partial\Gamma_n}{\partial p_z}\frac{\partial\Gamma_m}{\partial z} \tag{3.236}$$

But to obtain complex invariant one has to find expression for PB in complex form. for that we have to follow the eq.(3.233), from where one can easily get the following expression as

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_4}; \qquad \frac{\partial}{\partial p_x} = \frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3}; \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4}; \qquad \frac{\partial}{\partial p_y} = \frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}; \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial x_3} - i\frac{\partial}{\partial p_6}; \qquad \frac{\partial}{\partial p_z} = \frac{\partial}{\partial p_3} - i\frac{\partial}{\partial x_6}. \end{aligned} \tag{3.237}$$

The Hamilton's equations of motion for complex H, eq.(**??**), can be written as

$$\begin{aligned} \dot{x}_1 &= \frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_4}; \qquad \dot{p}_4 = \frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_4}; \\ \dot{x}_2 &= \frac{\partial H_1}{\partial p_2} + \frac{\partial H_2}{\partial x_5}; \qquad \dot{p}_5 = \frac{\partial H_2}{\partial p_2} - \frac{\partial H_1}{\partial x_5}; \\ \dot{x}_3 &= \frac{\partial H_1}{\partial p_3} + \frac{\partial H_2}{\partial x_6}; \qquad \dot{p}_6 = \frac{\partial H_2}{\partial p_2} - \frac{\partial H_1}{\partial x_6}. \end{aligned} \tag{3.238}$$

If the H, eq.(**??**), is an analytic function of complex variables, then H_1 and H_2 satisfy the Cauchy-Riemann conditions and after invoking such analyticity conditions, eq.(3.238) reduces

$$\dot{x}_1 = 2\frac{\partial H_1}{\partial p_1};\ \dot{p}_1 = -2\frac{\partial H_1}{\partial x_1};\ \dot{x}_2 = 2\frac{\partial H_1}{\partial p_2};\ \dot{p}_2 = -2\frac{\partial H_1}{\partial x_2};$$
$$\dot{x}_3 = 2\frac{\partial H_1}{\partial p_3};\ \dot{p}_3 = -2\frac{\partial H_1}{\partial x_3};\ \dot{x}_4 = 2\frac{\partial H_1}{\partial p_4};\ \dot{p}_4 = -2\frac{\partial H_1}{\partial x_4};$$
$$\dot{x}_5 = 2\frac{\partial H_1}{\partial p_5};\ \dot{p}_5 = -2\frac{\partial H_1}{\partial x_5};\ \dot{x}_6 = 2\frac{\partial H_1}{\partial p_6};\ \dot{p}_6 = -2\frac{\partial H_1}{\partial x_6}. \tag{3.239}$$

Note that $(x_1, p_1), (x_2, p_2), (x_3, p_3), (x_4, p_4), (x_5, p_5)$, and (x_6, p_6) constitute canonical pairs.

Now consider a complex phase space function $I(x, y, z, p_x, p_y, p_z, t)$ as $I = I_1 + iI_2$ Thus for function I to be the TD dynamical invariant of the system in complex phase space, then this must conform the following invariance condition:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H] = 0, \tag{3.240}$$

where $[.,.]$ is the Poisson bracket, which in view of the definition, eq.(3.233), turns out to be

$$\begin{aligned} [A, B]_{(x,p)} \quad &= [A, B]_{(x_1,p_1)} - i[A, B]_{(x_1,x_4)} - i[A, B]_{(p_4,p_1)} - [A, B]_{(p_4,x_4)} \\ &+[A, B]_{(x_2,p_2)} - i[A, B]_{(x_2,x_5)} - i[A, B]_{(p_5,p_2)} - [A, B]_{(p_5,x_5)}, \\ &+[A, B]_{(x_3,p_3)} - i[A, B]_{(x_3,x_6)} - i[A, B]_{(p_6,p_3)} - [A, B]_{(p_6,x_6)}. \end{aligned} \tag{3.241}$$

which indicates that the computation of Poisson bracket in case of complex Hamiltonian systems becomes a bit tedious.

3.3.1 Example

Consider a simple harmonic oscillator in three dimensions [20], whose Hamiltonian is given by

$$H = \frac{1}{2}[p_x^2 + p_y^2 + p_y^2] + \frac{w^2}{2}[x^2 + y^2 + z^2]. \tag{3.242}$$

Using eq.(3.233), the above Hamiltonian can be expressed as

$$\begin{aligned} H &= \frac{1}{2}p_1^2 - \frac{1}{2}x_4^2 + \frac{1}{2}p_2^2 - \frac{1}{2}x_5^2 + \frac{1}{2}p_3^2 - \frac{1}{2}x_6^2 + i[p_1x_4 + p_2x_5 + p_3x_6 + \omega^2(x_1p_4 + x_2p_5 + x_3p_6)] \\ &+\omega^2\left(\frac{1}{2}x_1^2 - \frac{1}{2}p_4^2 + \frac{1}{2}x_2^2 - \frac{1}{2}p_5^2 + \frac{1}{2}x_3^2 - \frac{1}{2}p_6^2\right) \\ &= \sum_{m=1}^{18} h_m(t)\Gamma_m(x_1, p_4, x_2, p_5, x_3, p_6, p_1, x_4, p_2, x_5, p_3, x_6),. \end{aligned} \tag{3.243}$$

and the various Γ's and h(t)'s for the above complex H are given as

$$\begin{aligned} &\Gamma_1 = \frac{1}{2}p_1^2; \quad \Gamma_2 = \frac{1}{2}x_4^2; \quad \Gamma_3 = \frac{1}{2}p_2; \quad \Gamma_4 = \frac{1}{2}x_5^2; \quad \Gamma_5 = \frac{1}{2}p_3^2; \quad \Gamma_6 = \frac{1}{2}x_6^2; \\ &\Gamma_7 = p_1x_4; \quad \Gamma_8 = p_2x_5; \quad \Gamma_9 = x_6p_3; \quad \Gamma_{10} = x_1p_4; \quad \Gamma_{11} = x_2p_5; \quad \Gamma_{12} = x_3p_4; \\ &\Gamma_{13} = \frac{1}{2}x_1^2; \quad \Gamma_{14} = \frac{1}{2}p_4; \quad \Gamma_{15} = \frac{1}{2}x_2; \quad \Gamma_{16} = \frac{1}{2}p_5; \quad \Gamma_{17} = \frac{1}{2}x_3^2; \quad \Gamma_{18} = \frac{1}{2}p_6^2. \end{aligned} \tag{3.244}$$

with

$$h_1 = 1, \quad h_2 = -1, \quad h_3 = 1, \quad h_4 = 1, \quad h_5 = 1, \quad h_6 = 1,$$
$$h_7 = i, \quad h_8 = i, \quad h_9 = i, \quad h_{10} = iw^2, \quad h_{11} = iw^2, \quad h_{12} = iw^2,$$
$$h_{13} = w^2, \quad h_{14} = -w^2, \quad h_{15} = w^2, \quad h_{16} = -w^2, \quad h_{17} = w^2, \quad h_{18} = -w^2. \quad (3.245)$$

The dynamical algebra in this case is not closed. To find closure property for the above system, we add twelve more phase space functions (Γ_l)'s. The additional (Γ_l)'s are as follow

$$\Gamma_{19} = p_6x_6; \quad \Gamma_{20} = p_3x_3; \quad \Gamma_{21} = p_3p_6; \quad \Gamma_{22} = x_3x_6; \quad \Gamma_{23} = x_5p_5; \quad \Gamma_{24} = x_2x_5;$$
$$\Gamma_{25} = x_1x_4; \; \Gamma_{26} = p_2x_2; \; \Gamma_{27} = p_2p_5; \; \Gamma_{28} = x_4p_4; \; \Gamma_{29} = p_1p_4; \; \Gamma_{30} = p_1x_1. \quad (3.246)$$

with corresponding $h_l(t) = 0$. Now in the light of Poisson bracket for complex systems, eq.(3.241), we get large number of non-vanishing Poisson brackets, namely

$$[\Gamma_1, \Gamma_{10}] = i\Gamma_{30} - \Gamma_{29}; \quad [\Gamma_1, \Gamma_{13}] = -\Gamma_{30}; \; [\Gamma_1, \Gamma_{14}] = i\Gamma_{29}; \quad [\Gamma_1, \Gamma_{25}] = -\Gamma_7;$$
$$[\Gamma_1, \Gamma_{28}] = i\Gamma_7; \quad [\Gamma_1, \Gamma_{29}] = 2i\Gamma_1; \; [\Gamma_1, \Gamma_{30}] = -2i\Gamma_1; \quad [\Gamma_2, \Gamma_{13}] = i\Gamma_{25};$$
$$[\Gamma_2, \Gamma_{10}] = \Gamma_{25} + i\Gamma_{28}; \quad [\Gamma_2, \Gamma_{14}] = \Gamma_{28}; \quad [\Gamma_2, \Gamma_{25}] = 2i\Gamma_2; [\Gamma_2, \Gamma_{28}] = 2\Gamma_2;$$
$$[\Gamma_2, \Gamma_{29}] = \Gamma_7; \quad [\Gamma_2, \Gamma_{30}] = i\Gamma_7; \; [\Gamma_3, \Gamma_{11}] = -\Gamma_{27} + i\Gamma_{26}; \; [\Gamma_3, \Gamma_{15}] = -\Gamma_{26};$$
$$[\Gamma_3, \Gamma_{16}] = i\Gamma_{27}; \quad [\Gamma_3, \Gamma_{23}] = i\Gamma_8; \; [\Gamma_3, \Gamma_{24}] = \Gamma_8; \quad [\Gamma_3, \Gamma_{26}] = -2\Gamma_3; \; [\Gamma_3, \Gamma_{27}] = 2i\Gamma_3;$$
$$[\Gamma_4, \Gamma_{11}] = i\Gamma_{23} + \Gamma_{24}; \quad [\Gamma_4, \Gamma_{15}] = \Gamma_{24} + i\Gamma_{23}; \quad [\Gamma_4, \Gamma_{16}] = \Gamma_{23};$$
$$[\Gamma_4, \Gamma_{23}] = 2\Gamma_4; \quad [\Gamma_4, \Gamma_{24}] = 2i\Gamma_4; \; [\Gamma_4, \Gamma_{26}] = i\Gamma_8; \quad [\Gamma_4, \Gamma_{27}] = -\Gamma_8;$$
$$[\Gamma_5, \Gamma_{12}] = -\Gamma_{21} + i\Gamma_{20}; \quad [\Gamma_5, \Gamma_{17}] = -\Gamma_{20}; \quad [\Gamma_5, \Gamma_{18}] = i\Gamma_{21};$$
$$[\Gamma_5, \Gamma_{19}] = +i\Gamma_9; \quad [\Gamma_5, \Gamma_{20}] = -2\Gamma_5; \; [\Gamma_5, \Gamma_{21}] = 2i\Gamma_5; \quad [\Gamma_5, \Gamma_{22}] = -\Gamma_9;$$
$$[\Gamma_6, \Gamma_{12}] = i\Gamma_{19} + \Gamma_{22}; \; [\Gamma_6, \Gamma_{17}] = i\Gamma_{18} - \Gamma_{20}; [\Gamma_6, \Gamma_{18}] = \Gamma_{19};$$
$$[\Gamma_6, \Gamma_{19}] = 2\Gamma_6; \quad [\Gamma_6, \Gamma_{20}] = i\Gamma_9; \; [\Gamma_6, \Gamma_{21}] = \Gamma_9; \quad [\Gamma_6, \Gamma_{22}] = 2\Gamma_6;$$
$$[\Gamma_7, \Gamma_{10}] = i\Gamma_{29} + \Gamma_{30} - \Gamma_{28} + i\Gamma_{25}; \quad [\Gamma_7, \Gamma_{13}] = \Gamma_{30} - i\Gamma_{25}; \; [\Gamma_7, \Gamma_{14}] = i\Gamma_{28} + \Gamma_{29};$$
$$[\Gamma_7, \Gamma_{25}] = i\Gamma_7 - 2\Gamma_2; \quad [\Gamma_7, \Gamma_{28}] = 2i\Gamma_2 + \Gamma_7; \quad [\Gamma_7, \Gamma_{29}] = 2\Gamma_1 + i\Gamma_7; \; [\Gamma_7, \Gamma_{30}] = -\Gamma_7 + 2i\Gamma_1;$$
$$[\Gamma_8, \Gamma_{11}] = i\Gamma_{24} + \Gamma_{26} - \Gamma_{23} + i\Gamma_{27}; \quad [\Gamma_8, \Gamma_{15}] = -\Gamma_{24} + i\Gamma_{26}; \; [\Gamma_8, \Gamma_{16}] = \Gamma_{23} + \Gamma_{27},$$
$$[\Gamma_8, \Gamma_{23}] = \Gamma_8 + 2i\Gamma_4; \quad [\Gamma_8, \Gamma_{24}] = i\Gamma_8 - 2\Gamma_4; \; [\Gamma_8, \Gamma_{26}] = -\Gamma_8 + 2i\Gamma_3; \quad [\Gamma_8, \Gamma_{27}] = i\Gamma_8 + 2\Gamma_3;$$
$$[\Gamma_9, \Gamma_{12}] = i\Gamma_{21} + \Gamma_{20} - \Gamma_{19} + i\Gamma_{22}; \quad [\Gamma_9, \Gamma_{17}] = -\Gamma_{22} + i\Gamma_{20}; \; [\Gamma_9, \Gamma_{18}] = i\Gamma_{19} + \Gamma_{21},$$
$$[\Gamma_9, \Gamma_{19}] = 2i\Gamma_6 + \Gamma_9; \quad [\Gamma_9, \Gamma_{20}] = -\Gamma_9 + 2i\Gamma_5; \quad [\Gamma_9, \Gamma_{21}] = 2\Gamma_5 + i\Gamma_9; \; [\Gamma_9, \Gamma_{22}] = i\Gamma_9 - 2\Gamma_6;$$
$$[\Gamma_{10}, \Gamma_{25}] = -i\Gamma_{10} - 2\Gamma_3; \quad [\Gamma_{10}; \Gamma_{28}] = +2i\Gamma_{14} - \Gamma_{10}; \; [\Gamma_{10}, \Gamma_{29}] = 2\Gamma_{14} - i\Gamma_{10};$$
$$[\Gamma_{10}, \Gamma_{30}] = -2i\Gamma_{13} + \Gamma_{10}; [\Gamma_{11}, \Gamma_{23}] = -2i\Gamma_{16} - \Gamma_{11}; \quad [\Gamma_{11}; \Gamma_{24}] = -i\Gamma_{18} - 2\Gamma_{15};$$
$$[\Gamma_{11}, \Gamma_{26}] = -2i\Gamma_{15} + \Gamma_{11}; \quad [\Gamma_{11}, \Gamma_{27}] = 2\Gamma_{16} - i\Gamma_{11}; \quad [\Gamma_{12}, \Gamma_{19}] = -2i\Gamma_{18} - \Gamma_{12};$$
$$[\Gamma_{12}, \Gamma_{20}] = \Gamma_{12} - 2i\Gamma_{17}; \quad [\Gamma_{12}, \Gamma_{21}] = 2\Gamma_{18} - i\Gamma_{12}, \quad [\Gamma_{12}, \Gamma_{22}] = -i\Gamma_{12} - 2\Gamma_{17};$$
$$[\Gamma_{13}, \Gamma_{25}] = -2i\Gamma_{13}; \quad [\Gamma_{13}, \Gamma_{28}] = -i\Gamma_7; \quad [\Gamma_{13}, \Gamma_{29}] = \Gamma_{10}; \quad [\Gamma_{13}, \Gamma_{30}] = 2\Gamma_{13};$$
$$[\Gamma_{14}, \Gamma_{25}] = -\Gamma_{17}; \quad [\Gamma_{14}, \Gamma_{28}] = -2\Gamma_{14}; \quad [\Gamma_{14}, \Gamma_{29}] = -2i\Gamma_{14}; \quad [\Gamma_{14}, \Gamma_{30}] = -i\Gamma_{10};$$
$$[\Gamma_{15}, \Gamma_{23}] = 2i\Gamma_8; \quad [\Gamma_{15}, \Gamma_{24}] = -2i\Gamma_{15}; \quad [\Gamma_{15}, \Gamma_{26}] = 2i\Gamma_{15};$$
$$[\Gamma_{15}, \Gamma_{27}] = \Gamma_{18}; \quad [\Gamma_{16}, \Gamma_{23}] = 2\Gamma_{16}; \quad [\Gamma_{16}, \Gamma_{26}] = 2\Gamma_{15}; \quad [\Gamma_{16}, \Gamma_{27}] = -2i\Gamma_{16};$$
$$[\Gamma_{17}, \Gamma_{19}] = -i\Gamma_{12}; \quad [\Gamma_{17}, \Gamma_{20}] = 2\Gamma_{17}; \quad [\Gamma_{17}, \Gamma_{21}] = \Gamma_{12}; \quad [\Gamma_{17}, \Gamma_{22}] = -2i\Gamma_{17};$$
$$[\Gamma_{18}, \Gamma_{19}] = -2\Gamma_{18}; \quad [\Gamma_{18}, \Gamma_{20}] = -i\Gamma_{12} \quad [\Gamma_{18}, \Gamma_{21}] = -2i\Gamma_{18}; \quad [\Gamma_{18}, \Gamma_{22}] = -\Gamma_{12} \quad (3.247)$$

Therefore, their use in eq.(1.11) yields the following set of Partial differential equations in λ's are :

$$\dot{\lambda}_1 = 4(i\lambda_{29} - \lambda_{30}); \qquad \dot{\lambda}_2 = -4(i\lambda_{25} + \lambda_{28}), \quad (3.248)$$

$$\dot{\lambda}_3 = 4(i\lambda_{27} - \lambda_{26}); \qquad \dot{\lambda}_4 = 4(i\lambda_{24} - \lambda_{23}), \quad (3.249)$$

$$\dot{\lambda}_5 = 4(i\lambda_{21} - \lambda_{22}); \qquad \dot{\lambda}_6 = -4(i\lambda_{22} + \lambda_{19}), \tag{3.250}$$

$$\dot{\lambda}_7 = 2(-\lambda_{25} + i\lambda_{28} - \lambda_{29} - i\lambda_{30}); \qquad \dot{\lambda}_8 = 2(i\lambda_{23} + \lambda_{24} - i\lambda_{26} - \lambda_{27}), \tag{3.251}$$

$$\dot{\lambda}_9 = 2(i\lambda_{19} - i\lambda_{20} - \lambda_{22} - \lambda_{21}); \qquad \dot{\lambda}_{10} = 2\omega^2(\lambda_{25} - i\lambda_{28} + \lambda_{29} + i\lambda_{30}), \tag{3.252}$$

$$\dot{\lambda}_{11} = 2\omega^2(i\lambda_{23} - \lambda_{24} + -i\lambda_{26} + \lambda_{16}); \qquad \dot{\lambda}_{12} = 2\omega^2(-i\lambda_{19} + i\lambda_{20} + \lambda_{21} + \lambda_{22}), \tag{3.253}$$

$$\dot{\lambda}_{13} = -4\omega^2(i\lambda_{25} - \lambda_{30}); \qquad \dot{\lambda}_{14} = 4\omega^2(i\lambda_{29} + \lambda_{28}), \tag{3.254}$$

$$\dot{\lambda}_{15} = -4\omega^2(\lambda_{24} - \lambda_{26}); \qquad \dot{\lambda}_{16} = -4\omega^2(\lambda_{23} - i\lambda_{27}), \tag{3.255}$$

$$\dot{\lambda}_{17} = 4\omega^2(\lambda_{20} - i\lambda_{22}); \qquad \dot{\lambda}_{18} = 4\omega^2(\lambda_{19} + i\lambda_{21}), \tag{3.256}$$

$$\dot{\lambda}_{19} = 2\omega^2(\lambda_6 + i\lambda_9 - i\lambda_{12} + \lambda_{18}), \tag{3.257}$$

$$\dot{\lambda}_{20} = 2\omega^2(\lambda_5 - i\lambda_9 + i\lambda_{12} - \lambda_{17}), \tag{3.258}$$

$$\dot{\lambda}_{21} = 2\omega^2(i\lambda_5 + \lambda_9 - \lambda_{12} + i\lambda_{18}), \tag{3.259}$$

$$\dot{\lambda}_{22} = 2\omega^2(-i\lambda_6 + \lambda_9 - \lambda_{12} - i\lambda_{18}), \tag{3.260}$$

$$\dot{\lambda}_{23} = 2\omega^2(\lambda_4 + i\lambda_8 - i\lambda_{11} - \lambda_{16}), \tag{3.261}$$

$$\dot{\lambda}_{24} = 2\omega^2(-i\lambda_4 + \lambda_8 - \lambda_{11} - i\lambda_{15}), \tag{3.262}$$

$$\dot{\lambda}_{25} = 2\omega^2(\lambda_7 - i\lambda_2 - \lambda_{11} - i\lambda_{15}), \tag{3.263}$$

$$\dot{\lambda}_{26} = 2\omega^2(\lambda_4 - i\lambda_8 + i\lambda_{11} - \lambda_{15}). \tag{3.264}$$

$$\dot{\lambda}_{27} = 2\omega^2(i\lambda_3 + \lambda_8 - \lambda_{11} + i\lambda_{16}), \tag{3.265}$$

$$\dot{\lambda}_{28} = 2\omega^2(i\lambda_7 + \lambda_2 - i\lambda_{10} - \lambda_{14}), \tag{3.266}$$

$$\dot{\lambda}_{29} = 2\omega^2(i\lambda_1 + \lambda_7 - \lambda_{10} + i\lambda_{14}), \tag{3.267}$$

$$\dot{\lambda}_{30} = 2\omega^2(\lambda_1 + i\lambda_7 + i\lambda_{10} - \lambda_{13}). \tag{3.268}$$

In fact, to solve these 30 coupled PDEs for complex λ's is very difficult. Thus, here we make certain choices for λ's which facilitate to find solutions of above equations.

From eqs. (3.248) and (3.250), we get $2i\dot{\lambda}_7 = \dot{\lambda}_2 - \dot{\lambda}_1$, and if we consider $\lambda_7 = c_7$ (a constant), further $\lambda_1 = \lambda_2 = \eta(t)$, which immediately gives

$$\lambda_1 = \eta(t) + c_1; \qquad \lambda_2 = \eta(t) + c_2. \tag{3.269}$$

From eqs.(3.249) and (3.251), we get $2i\dot{\lambda}_8 = \dot{\lambda}_4 - \dot{\lambda}_3$, and if we consider $\lambda_8 = c_8$ (a constant), further $\lambda_4 = \lambda_3 = \theta(t)$, which immediately gives

$$\lambda_4 = \theta(t) + c_4; \qquad \lambda_3 = \theta(t) + c_3. \tag{3.270}$$

Again from eqs. (3.250) and (3.252), we get $2i\dot{\lambda}_9 = \dot{\lambda}_6 - \dot{\lambda}_5$, and if we consider $\lambda_9 = c_9$ (a constant), further $\lambda_6 = \lambda_5 = \zeta(t)$, which immediately gives

$$\lambda_5 = \zeta(t) + c_5; \qquad \lambda_6 = \zeta(t) + c_6. \tag{3.271}$$

From eqs.(3.254) and (3.252), we get $2i\dot{\lambda}_{10} = -\dot{\lambda}_{13} + \dot{\lambda}_{14}$, and if we consider $\lambda_{10} = c_{10}$ (a constant), further $\lambda_{13} = \lambda_{14} = \delta(t)$, which immediately gives

$$\lambda_{13} = \delta(t) + c_{13}; \qquad \lambda_{14} = \delta(t) + c_{14}. \tag{3.272}$$

Again from eqs. (3.255) and (3.253), we get $2i\dot{\lambda}_{11} = \dot{\lambda}_{16} + \dot{\lambda}_{15}$, and if we consider $\lambda_{11} = c_{11}$ (a constant), further $-\lambda_{16} = \lambda_{15} = \nu(t)$, which immediately gives

$$\lambda_{15} = \nu(t) + c_{15}; \qquad \lambda_{16} = -\nu(t) + c_{16}. \tag{3.273}$$

From eqs.(3.256) and (3.253), we get $2i\dot{\lambda}_{12} = \dot{\lambda}_{18} - \dot{\lambda}_{17}$, and if we consider $\lambda_{12} = c_{12}$ (a constant), further $\lambda_{17} = \lambda_{18} = \xi(t)$, which immediately gives

$$\lambda_{17} = \xi(t) + c_{17}; \qquad \lambda_{18} = \xi(t) + c_{18}. \tag{3.274}$$

Now for finding the solutions of λ_{19} and λ_{22}, subtract i times eq.(3.260) from eq.(3.257) and after using eq.(3.271), we get

$$\dot{\lambda}_{22} - i\dot{\lambda}_{19} = 2i(\lambda_{17} + \lambda_{18})$$

or

$$i\dot{\lambda}_{22} + \dot{\lambda}_{19} = -2(2\xi + c_{17} + c_{18}). \tag{3.275}$$

On the other hand, time derivative of eq.(3.250) is written as

$$\ddot{\lambda}_6 = 4(-i\lambda_{17} + \lambda_{18}) = \ddot{\phi}. \tag{3.276}$$

Hence using eq.(3.275) and (3.276), one immediately get

$$\lambda_{19} = -\frac{1}{8}(\dot{\phi} - 8\sigma_1) + c_{19}. \tag{3.277}$$

and

$$\lambda_{22} = \frac{i}{8}(\dot{\phi} - 8\sigma_1) + c_{22}, \tag{3.278}$$

where $\sigma_1 = \int(2\xi(t) + c_{17} + c_{18})dt$.

Similarly, from eq.(3.259-3.268), we obtain solutions for $(\lambda_{20} - \lambda_{30})$ respectively as

$$\lambda_{20} = -\frac{i}{8}(\dot{\phi} - 8\sigma_1) + c_{20}. \tag{3.279}$$

$$\lambda_{21} = \frac{1}{8}(\dot{\phi} + 8\sigma_1) + c_{21}. \tag{3.280}$$

$$\lambda_{23} = \frac{i}{8}(\dot{\chi} - 8\sigma_2) + c_{23}. \tag{3.281}$$

$$\lambda_{24} = \frac{1}{8}(\dot{\chi} + 8\sigma_2) + c_{24}., \tag{3.282}$$

$$\lambda_{25} = \frac{1}{8}(\dot{\psi} + 8\sigma_3) + c_{25}. \tag{3.283}$$

$$\lambda_{26} = -\frac{i}{8}(\dot{\chi} - 8\sigma_2) + c_{26}. \tag{3.284}$$

$$\lambda_{27} = \frac{1}{8}(\dot{\chi} + 8\sigma_2) + c_{27}. \tag{3.285}$$

$$\lambda_{28} = \frac{i}{8}(\dot{\psi} - 8\sigma_3) + c_{28}. \tag{3.286}$$

$$\lambda_{29} = \frac{1}{8}(\dot{\xi} + 8\sigma_3) + c_{29}. \tag{3.287}$$

$$\lambda_{30} = \frac{1}{8}(\dot{\psi} + 8\sigma_3) + c_{30}. \tag{3.288}$$

where $\sigma_2 = \int(2\nu(t) + c_{15} + c_{16})dt$ and $\sigma_3 = \int(2\delta(t) + c_{13} + c_{14})dt$.

Solutions from eqs.[(3.248)-(3.268)] in terms of arbitrary functions $\phi, \chi, \psi, \delta, \nu, \xi$ contains a lot of complex constants, c_i's, $(i = 1, ..., 30)$. If one put back these solutions for λ_i, $(i = 1, ..., 30)$ in eqs.[(3.248)-(3.268)], we obtain a number of restraint among these c_i's, and $\phi, \chi, \psi, \delta, \nu, \xi$, which limit the choices of these arbitrary complex quantities. If we set all $c_i's = 0$, then these relations are written as

$$\ddot{\phi} + 16\omega^2\phi = 0;\ \ddot{\chi} + 16\omega^2\chi = 0;\ \ddot{\psi} + 16\omega^2\psi = 0;$$
$$\ddot{\delta} + 16\omega^2\delta = 0;\ \ddot{\nu} + 16\omega^2\nu = 0;\ \ddot{\xi} + 16\omega^2\xi = 0. \tag{3.289}$$

If we set all c_i's equal to zero, then the solutions (for $\omega = 1$) to these equations can be written as

$$\phi(t) = \chi(t) = \psi(t) = \delta(t) = \nu(t) = \xi(t) = e^{4t}.$$

Therefore, after substituting the solutions of λ_i's in the eq.(3.66), the complex invariant for a three-dimensional complex oscillator becomes

$$\begin{aligned} I &= \frac{1}{2}\eta(p_1^2 + x_4^2) + \frac{1}{2}\theta(p_2^2 + x_5^2) + \frac{1}{2}\zeta(p_3^2 + x_6^2) + \frac{1}{2}\delta(x_1^2 + p_4^2) + \frac{1}{2}\nu(x_2^2 + p_5^2) + \frac{1}{2}\xi(x_3^2 + p_6^2) \\ &+ \frac{i}{8}(\dot{\phi} - 8\sigma_1)(x_3x_6 - p_3x_6) - \frac{1}{8}(\dot{\phi} + 8\sigma_1)(x_3p_3 + x_6p_6) - \frac{i}{8}(\dot{\chi} - 8\sigma_2)(x_2x_5 - p_2p_5) \\ &- \frac{1}{8}(\dot{\chi} + 8\sigma_2)(p_2x_2 + p_5x_5) + \frac{i}{8}(\dot{\xi} - 8\sigma_3)(x_1x_4 - p_1p_4) - \frac{1}{8}(\dot{\xi} + 8\sigma_3)(p_4x_4 + p_1x_1). \end{aligned} \tag{3.290}$$

which conforms to condition eq.(3.66) in view of the PB.

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Chapter 4

Invariants for Classical and Quantum Dynamical Systems

In classical mechanics, integrable systems are of interest, because they have regular trajectories. Indeed, their motion is restricted to a torus in phase space. Superintegrable systems are even more regular. Trajectories are completely determined by the values of the $2n - 1$ integrals of motion. In particular, all bounded trajectories are periodic, as in the case of the harmonic oscillator, or Kepler problem. In quantum mechanics, integrability, i.e., the existence of n integrals of motion, provides a complete set of quantum numbers, characterizing the system. Moreover, it simplifies the calculation of energy levels and wave functions. Superintegrability, in all cases studied so far, bring about exact solvability. This means that energy levels in superintegrable systems can be calculated algebraically, i.e., they satisfy algebraic rather than transcendental equations.

As it has been already mention in the thesis that, there have been considerable efforts in exploring the method of construction of these invariants for both time independent and time dependent systems in recent years. Although several attempts have been made for the study of one and higher dimensional systems. However, the study of higher dimensional system seems to be rather more difficult for Hamiltonian system. Here the study of the invariants in higher order, particularly for two dimensional time independent system (TID) and time dependent system (TD) is carried out. In chapter number one different methods for construction of invariants are discussed. For the construction of invariants or constants of motion, there have been many mathematical methods developed in the past. But there is no such method which have universal character, and in most of the cases to obtain concrete results one or more adhoc assumptions are to be made. Here in thesis, we describe only a few methods for construction of invariants of desired order for both classical and quantum dynamical systems.

4.1 Classical Systems

Many situations arise in different branches of theoretical science where the role of real and complex Hamiltonian becomes desirable, but only a few complex systems have been identified which have their role in explaining the physical phenomenon (In most of them only the momentum becomes complex). Complex invariants, corresponding to Hamiltonian have been searched by many authors [3, 4] but their physical significance have been highlighted in some branches of physics. Some complex invariants have been searched by Jarmo Heitarienta in which integrable hamiltonian systems of type

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V(x, y) \tag{4.1}$$

where V is a polynomial in x and y of degree 5 or less are taken (both homogeneous and non-homogeneous). The second invariant are found to be of a polynomial in p_x and p_y of order 4 or less. Complex invariants that are linear in position and momentum have also been discussed [3] for a TD classical quadratic oscillator. Each pair of (real and complex) invariants realize explicitly a canonical transformation from the phase space to the invariant space, in which the action-phase variables are defined.

Even though it is mentioned in section 2.4 of Chapter two that the classical invariant in two dimension of fourth order are obtained but choice of potential may trivialize the problem. Obtaining invariant for nonseparable potential as a new material for the thesis may be possible, but it is beyond the scope of thesis. Invariants corresponding to these potentials are possible, if we make the coordinate transformation for Hamiltonian into spherical polar coordinate and try to calculate the corresponding invariants for both classical and quantum dynamical systems. However we considered some potentials which are nonseparable such as coulomb potential, kepler potential and their combination to diversify the problem and to obtain new result.

We tried to calculate the invariant for the potential that are nonseparable, but we found the calculation going to be very tedious and did not work out in the present formulation.

However the choice of potential that are separable will not trivialize the problem. To support this claim, we can observe that following systems studied by Fokas [1]. A nonseparable Hamiltonian of form

$$H = \frac{p_x^2 + p_y^2}{2} + (x^2 - y^2)^{-\frac{2}{3}} \tag{4.2}$$

and invariant for this system is given by

$$I = (p_x^2 + p_y^2)(xp_y - yp_x) - 4(yp_x + xp_y)(x^2 - y^2)^{-\frac{2}{3}}$$

Simultaneously he studied another separable form of Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2} + Ax^2 + By^2 \tag{4.3}$$

has the invariant for this system is given by

$$I = (yp_x^2 - xp_y^2)^2 + (B - A)p_x^2 + 2x^4 + 2x^2y^2 + 2Ax^2$$

Here we can observe that though the Hamiltonian is separable in one dimension but the the invariant obtained is in two dimension i.e. non-trivial case. Hietarinta [2] studied well known Hamiltonian of form

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2}{2} + \frac{y^2}{18} \tag{4.4}$$

And invariant for this system is given by

$$I = p_y^2(xp_y - yp_x) + \frac{y^3 p_x}{27} - \frac{xy^2 p_y}{3}$$

Here we can observe that though the Hamiltonian is separable in one dimension but the the invariant obtained is in two dimension i.e. non-trivial case. To conform further we take example of Toda type Hamiltonian is given by

$$H = \frac{p_x^2 + p_y^2}{2} + e^y + e^{x-y} \tag{4.5}$$

And invariant for this system is given by

$$I = p_x^2 p_y^2 + 2e^y p_x^2 - 2e^{x-y} p_x p_y + e^{2x-2y} + 2e^x$$

Again Here we can observe the invariant obtained is in two dimension i.e. non-trivial case. Hamiltonian of form

$$H = \frac{p_x^2 + p_y^2}{2} + (x^2 - y^2)^2 \tag{4.6}$$

And invariant for this system is given by

$$I = yp_x - xp_y$$

4.2 Some Applications of Dynamical Invariant

The invariants when defined in a broader sense play an important role in the domains of a variety of fields. In particular, the invariants are capable of finding the solution of equation of motion, can reduce some nonlinear problems to a quadrature, to reduce the order of differential equations [3], for testing stability of differential equations (onset of chaos) [4] and to assess the accuracy of numerical simulations of TD systems [5]. Also the higher order invariants provide the internal symmetry of physical systems particularly in molecular dynamics [6].

In the following sections, we discuss some situations where the invariants have played their distinctive role in classical dynamical systems.

4.2.1 Applications to Classical Systems

Direct Validation Technique For Numerical Simulations

Computer simulations have become an indispensable tool for studies of the physics of dynamical systems. To achieve an appropriate level of confidence in the simulation results, accuracy assessments must be conceived as an integral part of the simulation strategies. A common indirect validation method is to cross-check the results of commensurable simulation codes against each other. Furthermore, the simulation results of particularly chosen scenarios may be compared against analytical models.

With the knowledge of invariant of time-dependent Hamiltonian systems for we present a direct technique for the error assessment of numerical simulations of time-dependent Hamiltonian systems. The method is based on an invariant I that has been shown to exist for the Hamiltonian systems with general time-dependent potentials [1, 2]. Because of the generally limited accuracy of numerical methods, this invariant I can never be realized strictly in numerical simulations. The relative deviation of a numerically calculated invariant $I(t)$ from the exact invariant $I_0 = I(0)$ may then be taken as the error estimation for the respective simulation. In this sense, the direct error assessment technique can be regarded as a generalization of a validation method that is applicable for autonomous (time-independent) Hamiltonian systems. For these cases, the Hamiltonian H itself represents an invariant. Accordingly, the relative deviation of $H(t)$ from H_0 in the simulation is commonly used as an accuracy criterion. We apply this approach to estimate the accuracy of a simulation of a one-dimensional Hamiltonian system. Now, we work out the invariant and the auxiliary equation in the particular form pertaining to this Hamiltonian system.

The method for construction of exact invariants for TD classical Hamiltonians systems is already discussed in section 1.5 of chapter 1.

Example. 1

1. Time-dependent damped asymmetric spring

As a simple example, we investigate the one-dimensional nonlinear system of a time-dependent damped asymmetric spring.Its Hamiltonian is defined by

$$H = \frac{1}{2}e^{-F(t)}p^2 + [\frac{\omega^2(t)}{2}x^2 + a(t)x^3]e^{F(t)}.$$

Writing $f(t) = \dot{F}(t)$, the equation of motion follows as

$$\dot{x} = pe^{-F(t)}, \quad \ddot{x} + f(t)\dot{x} + \omega^2(t)x + 3a(t)x^2 = 0. \tag{4.7}$$

The invariant I is immediately found using above method and writing the general invariant for one degree of freedom with the Hamiltonian H given by

$$I = \frac{1}{2}e^{2F(t)}[f_2\dot{x}^2 - \dot{f}_2 x\dot{x} + x^2\{\frac{1}{2}\ddot{f}_2 + \frac{1}{2}\dot{f}_2 f(t) + f_2\omega^2(t) + 2xf_2a(t)\}]. \tag{4.8}$$

The function $f_2(t)$ for this particular case is given as a solution of the linear third-order ordinary differential equation

$$\dddot{f} + 3\ddot{f}_2(t) + \dot{f}_2 f(t) + 2\dot{f}_2 f^2(t) + 4\dot{f}_2\omega^2(t) + 4f_2 f(t)\omega^2(t) + 4f_2\omega\dot{\omega}(t)$$
$$+2x(t)[2f_2\dot{a}(t)f(t) + 5\dot{f}_2 a(t)] = 0 \tag{4.9}$$

Since the particle trajectory $x = x(t)$ is explicitly contained in Eq. (4.9), the solution $f_2(t)$ can only be obtained integrating Eq.(4.9) simultaneously with the equation of motion (4.7). We may easily convince ourselves that I is indeed a conserved quantity. Calculating the total time derivative of Eq. (4.8), and inserting the equation of motion (4.7), we end up with Eq. (4.9), which is fulfilled by definition of $f_2(t)$ for the given trajectory $x = x(t)$.

The third-order equation (4.9) may be converted into a coupled set of first- and second-order equations. The second order equation

$$\ddot{f}_2^2 - \frac{\dot{f}_2^2}{2f_2} + \dot{f}_2 f(t) + 2f_2\omega^2(t) = \frac{g_x(t)}{f_2}e^{-2F(t)} \tag{4.10}$$

is equivalent to Eq. (4.9) if the time derivative of $g_x(t)$, introduced in Eq. (4.10), is given by

$$\dot{g}_x(t) = -2x(t)f_2 e^{2F(t)}(2f_2\dot{a} + 4f_2 af + 5\dot{f}_2 a). \tag{4.11}$$

With the help of the auxiliary equation (4.10), the invariant (4.8) may be expressed in the alternative form

$$I = \frac{e^{2F(t)}}{2f_2}\left[\left(f_2\dot{x} - \frac{1}{2}\dot{f}_2 x\right)^2 + 2x^3 f_2^2(t)a(t)\right] + \frac{g_x(t)}{4f_2}x^2. \tag{4.12}$$

The existence of a constant of motion I for general explicitly time-dependent Hamiltonian systems has been shown to be useful to check the accuracy of numerical simulations of this class of dynamical systems.

Example. 2 Time-dependent anharmonic undamped one-dimensional oscillator.

As a second example, we investigate the one-dimensional non-linear system of a time-dependent anharmonic oscillator without damping, defined by the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2(t)q^2 + a(t)q^3 + b(t)q^4 : \tag{4.13}$$

The associated equation of motion is given by

$$\dot{q} = p; \qquad \ddot{q} + \omega^2(t)q^2 + a(t)q^3 + b(t)q^4 = 0 : \tag{4.14}$$

Again, the invariant is immediately obtained writing the general invariant for one dimension.

$$I = \frac{1}{2}f(t)\left[\dot{q}^2 + \omega^2(t)q^2 + 2a(t)q^3 + 2b(t)q^4\right] - \frac{1}{2}\dot{f}(t)q\dot{q} + \frac{1}{4}\ddot{f}(t)q^2 \tag{4.15}$$

For this particular case, the linear third-order equation for the auxiliary function f(t) reads

$$\dddot{f} + 4\dot{f}\omega^2(t) + 4f\omega\dot{\omega} + \omega 2q(t)[2f\dot{a} + 5\dot{f}a] + 4q^2(t)[f\dot{b} + 3\dot{f}b] = 0 \tag{4.16}$$

which follows from the general form of Eq. (4.9). We observe that . in contrast to the previous linear example . the particle trajectory $q = q(t)$ is explicitly contained in in the related auxiliary Eq. (4.16). Consequently, the integral function $f(t)$ can only be determined if Eq. (4.16) is integrated simultaneously with the equation of motion (4.14). We may directly convince ourselves that I is indeed a conserved quantity. Calculating the total time derivative of Eq. (4.15), and inserting the equation of motion (4.14), we end up with Eq. (4.16), which is fulfilled by definition of f(t) for the given trajectory $q = q(t)$. The third-order differential Eq. (4.16) may be converted into a coupled set of first- and second-order equations. It is easily shown that the non-linear second-order equation

$$f\dot{f} - \frac{1}{2}f^2 + 2\omega 2(t)f^2 = g(t) \tag{4.17}$$

is equivalent to Eq. (4.16), provided that the time derivative of the function g(t), introduced in Eq. (4.16), is given by

$$\dot{g}(t) = -2q(t)f[2f\dot{a} + 5\dot{f}a] - 4q^2(t)f[f\dot{b} + 3\dot{f}b]. \tag{4.18}$$

With the help of the auxiliary equation in the form of Eq. (4.16), the invariant (4.16) may be expressed equivalently as

$$I = \frac{1}{2}\left[f\dot{q}^2 - \dot{f}q\dot{q} + \frac{\dot{f}^2}{4f}q^2 + 2faq^3 + 2fbq^4\right] + \frac{g(t)}{4f}q^2 \tag{4.19}$$

Having numerically integrated the equations of motion, the systems auxiliary equation can be numerically calculated, and the numerical value of the invariant $I(t)$ be obtained thereof. The relative deviation $\frac{\triangle I}{I_0}$ of $I(t)$ from the exact invariant I_0 defined by the initial conditions can then be used as a measure for the accuracy of the respective simulation. Comparing simulation runs with different parameters, such as the number of macro-particles, the time step size, details of the numerical algorithm used to integrate the equations of motion, we may straightforwardly check whether the overall accuracy of our particular simulation has been improved or not.

4.3 Quantum Systems

In the quantum case, this was shown distinctly by Lewis and Riesenfeld [9], whose method was generalized and applied to different problems in numerous publications. A quantum integral of motion is defined usually as an operator whose average value $\langle\psi(t)|\hat{I}(t)|\psi(t)\rangle$ does not depend on time for any state $|\psi(t)\rangle$ obeying the Schrödinger equation. $\hat{I}(t)$ satisfies the equation $i\hbar\partial\hat{I}/\partial t = [\hat{H}, \hat{I}]$, therefore its explicit form depends on the form of the Hamilton operator H (which is supposed to be Hermitian).

Let us now consider the quantum (SHO) system whose Hamiltonian is given as,

$$H = \frac{1}{2m}[p^2 + \omega(t)^2 x^2].$$

where p and x are now required to satisfy the commutation relation $[x, p] = i\hbar$.

We also take p to be real, which is possible if $\omega^2(t)$ is real. Using the commutation relation and the equation for p, it is easy to show that the quantity I which is an invariant of the classical system is also a quantum mechanical constant of the motion. That is, I satisfies

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{\hbar}[I, H] = 0.$$

Therefore, I has eigen states whose eigenvalues are time independent. These eigen states and eigenvalues of I can be found by a method that is completely analogous to the method introduced by Dirac for finding the eigen states and eigenvalues of the Hamiltonian of a harmonic oscillator. It seems natural to apply to the definition of classical integrability Dirac's prescription of replacing- Poisson brackets by commutators of corresponding quantum operators

$$\{\ ,\ \} \rightarrow \ \frac{1}{\hbar}[\ ,\ ]$$

in order to introduce the notion of quantum integrability.

For example, in the simplest cases of a quantum harmonic oscillator with a time-dependent frequency $\omega(t)$ or a charged particle moving in a time-dependent homogeneous magnetic field, one has linear integrals of motion of the form

$$A(t) = u(t)\hat{p} - \dot{u}(t)\hat{x}$$

where $u(t)$ is a solution to the classical equation $\ddot{u}(t) + \omega^2(t)u = 0$ (here $\hbar = m = 1$). Depending on the choice of the concrete solution $u(t)$, eigen states of the operator. $\hat{A}(t)$ may be either generalized coherent states, or squeezed correlated states, or propagators in various representations.

Integrability of a quantum system often manifests itself in the form of classical integrability for a related, usually discrete system. The question of whether quantum integrability is a consequence of classical integrability has been continual and is still unanswered in full generality. Against the belief that the classical integrability of a system implies its quantum integrability in a inconsequential manner, investigations [9, 10] have shown that this indeed is not the case and therefore classical integrable (CI) systems are not necessarily be quantum integrable (QI). The comparison between classical and quantum integrability becomes easier if we represent the quantum operators by c-numbers [11] because classical and quantum mechanics are not algebraic isomorphic. Differences are bound to appear somewhere even if one uses the c-numbers in both. Indeed, the commutator becomes the Moyal bracket [12], which reduces to Poisson bracket only when $\hbar \to 0$. It has been observed that if classical invariants are second order in momenta, a straightforward quantization be good enough to give commuting operators. However, not all invariants are p-quadratic and for some of them no linear quantization procedure can lead to commuting operators. It has turned out that, even in these cases, one can obtain quantum integrability by adding suitable quantum corrective terms [13, 14], which are explicitly $\hbar$ dependent, to both the invariant and Hamiltonian. In this chapter we intend to construct the quantum invariants of a set of potentials with second order quantum corrections $\mathrm{O}(\hbar^2)$ in two and three dimensions [15]. The correspondence rules are characterized by a function of $2n$ variables $\mathcal{F}(x,y)$. The formal integral used to demonstrate these correspondence rules between the c-number $A(p,q)$ and the quantum operator $\hat{A}(\hat{p},\hat{q})$ is given by

$$\hat{\mathcal{A}}(\hat{p},\hat{q}) = \int d^n q d^n p d^n x d^n y (2\pi\hbar)^{-2n} \mathcal{F}(x,y) A_{\mathcal{F}}(p,q) \exp[i\{x.(\hat{p}-p) + y.(\hat{q}-q)\}/\hbar], \quad (4.20)$$

where the label $\mathcal{F}$ to A indicates that the operator $\hat{\mathcal{A}}$ is fixed and $A_{\mathcal{F}}(p,q)$ will be different for different $\mathcal{F}(x,y)$. Hietarinta [9] described in his work as once the correspondence is given, the relation between the operators $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}$ in the form

$$[\hat{\mathcal{A}}, \hat{\mathcal{B}}] = i\hbar\hat{\mathcal{C}},$$

can be translated into the corresponding c-number representations A, B, C for Wingre-Weyl ordering rule $(F \equiv 1)$ as from the Moyal bracket the expansion of sine function.

The notion of this concept is more transparent at the classical level in the spirit of Liouville's theorem, at the levels of quantum mechanics and field theory, however it requires further investigations. The classical integrability of a system entails its quantum integrability in a trivial manner, investigations have shown that this indeed is not the case. It is already mentioned in chapter one, for constructing the quantum invariant of a system from the corresponding classical one, the quantum corrections arising from the terms involving $\hbar$ in the expansion of the sine function in eq.(1.2) need to be calculated. It can be seen from eq.(1.2) that in the lowest order in the power of $\hbar$, the Moyal bracket $\{A,B\}_{MB} = A \overset{\leftrightarrow}{\wedge} B$ which is just Poisson bracket $[A,B]_{PB}$. It is also mentioned that if the second invariant is at most of second order in momenta then Moyal bracket exactly reduces to the Poisson bracket and the classical and quantum invariant turn out to be identical. For cubic and higher order invariants however, the quantum corrections arise and only after these corrections a classical invariant becomes quantum invariant in the spirit of Moyal bracket. These corrections are obtained upto $\mathrm{O}(\hbar^2)$. Several authors [11, 16] have studied the comparison of Classical integrability and Quantum integrability. Their studies reveals the problem of quantum integrability with quantum corrections. Keeping in view we plan to study the higher order i.e. upto fourth order quantum invariant after making necessary quantum corrections to the corresponding classical invariant. In this section, we discuss the method to construct quantum invariant after making necessary second order corrections $\mathrm{O}(\hbar^2)$ to the corresponding classical invariant.

4.4 Quantum invariants in two dimensions

In the second chapter third order classical invariants have been worked out for two dimensions by using the rationalization method. Now we introduce Moyal's bracket to construct quantum invariants to the corresponding classical invariants with some corrections.

The method

For a dynamical system, the Hamiltonian is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(x_1, x_2). \tag{4.21}$$

The invariant of the system is obtained from $dI/dt = [I, H]_{PB} = 0$, where the Poisson bracket of n canonical coordinates and momenta is as follows

$$[x_i, p_j] = \delta_{ij},$$

whereas for the quantum case, we write

$$[\hat{x_i}, \hat{p_j}] = i\hbar\delta_{ij}, \qquad \hat{x}_i\psi = x_i\psi,$$

and

$$\hat{p}_j\psi = -i\hbar\frac{\partial\psi}{\partial x_j}.$$

Quantum invariant is obtained from the expression

$$\frac{dI}{dt} = \{I, H\}_{MB} = 0. \tag{4.22}$$

The Moyal bracket $\{A, B\}_{MB}$ for two functions A and B is given as

$$\begin{aligned} \{A, B\}_{MB} = &\quad (2/\hbar)A\sin((1/2)\hbar \overset{\leftrightarrow}{\wedge})B \\ = &\quad A \overset{\leftrightarrow}{\wedge} B - (1/24)\hbar^2 A \overset{\leftrightarrow}{\wedge}^3 B + (1/1920)\hbar^4 A \overset{\leftrightarrow}{\wedge}^5 B + ..., \end{aligned} \tag{4.23}$$

in which $\overset{\leftrightarrow}{\wedge}$ is given by

$$\overset{\leftrightarrow}{\wedge} = \sum_{i=1}^{n}\left[\frac{\overleftarrow{\partial}}{\partial x_i}\frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial x_i}\right].$$

For quantum integrability it is required that the Moyal bracket of H and I vanishes. Since invariant I is fourth order in p, described in eq.(2.235), then the second term in eq.(4.23) might contribute for quantum invariant. As Hamiltonian is order of p^2, therefore $\partial_p^3, \partial_p^2\partial_x$ and $\partial_p\partial_x^2$ all annihilate it and the only contribution comes out

$$-\frac{2}{\hbar}\frac{1}{3!}\left(\frac{\hbar}{2}\right)^3\left(\sum_i \frac{\partial}{\partial x_i}\frac{\partial}{\partial p_i}\right)^3 V(x_i)I(p_i, x_i), \tag{4.24}$$

Therefore, in quantum case the eqs.(2.172)-(4.11) remain unchanged, but eqs.(4.12) and (4.13) required to be modified for construction of fourth order invariants as

$$\partial_x f_9 = -\frac{\hbar^2}{4}[4f_1\partial_x^3 V + 3f_2\partial_x^2\partial_y V + 2f_3\partial_x\partial_y^2 V + f_4\partial_y^3 V]2f_6\partial_x V + f_7\partial_y V, \tag{4.25}$$

$$\partial_y f_9 = -\frac{\hbar^2}{4}[f_2\partial_x^3 V + 2f_3\partial_x^2\partial_y V + 3f_4\partial_x\partial_y^2 V + 4f_5\partial_y^3 V] + f_7\partial_x V + 2f_8\partial_y V, \tag{4.26}$$

If the Hamiltonian in eq.(4.22) is classically integrable and has the second invariant of type eq.(2.171) and satisfies the following conditions

$$4f_1\partial_x^3 V + 3f_2\partial_x^2\partial_y V + 2f_3\partial_x\partial_y^2 V + f_4\partial_y^3 V = 0, \tag{4.27}$$

$$f_2\partial_x^3 V + 2f_3\partial_x^2\partial_y V + 3f_4\partial_x\partial_y^2 V + 4f_5\partial_y^3 V = 0, \tag{4.28}$$

then the corresponding quantum Hamiltonian is also integrable without any quantum corrections. If eqs.(4.27) and (4.28) do not hold, it might be possible to solve for f_9 from eqs.(4.25) and (4.26), if the $\hbar^2$ terms satisfy the following integrability condition

$$\partial_y[4f_1\partial_x^3 V + 3f_2\partial_x^2\partial_y V + 2f_3\partial_x\partial_y^2 V + f_4\partial_y^3 V] = \partial_x[f_2\partial_x^3 V + 2f_3\partial_x^2\partial_y V + 3f_4\partial_x\partial_y^2 V + 4f_5\partial_y^3 V], \tag{4.29}$$

then, we write

$$f_9(Quantum) = f_9(classical) + \hbar^2\Delta_Q f_9, \tag{4.30}$$

So the quantum correction to I would be $\Delta_Q I = \hbar^2\Delta_Q f_9$.

Hence

$$I(Quantum) = I(Classical) + \hbar^2\Delta_Q f_9. \tag{4.31}$$

The coefficient f_9 can be solved from the following two equations

$$\partial_x\Delta_Q f_9 = -\frac{1}{4}[4f_1\partial_x^3 V + 3f_2\partial_x^2\partial_y V + 2f_3\partial_x\partial_y^2 V + f_4\partial_y^3 V], \tag{4.32}$$

$$\partial_y\Delta_Q f_9 = -\frac{1}{4}[f_2\partial_x^3 V + 2f_3\partial_x^2\partial_y V + 3f_4\partial_x\partial_y^2 V + 4f_5\partial_y^3 V]. \tag{4.33}$$

4.4.1 Illustrative examples

1. First example (Inverse square potential or coulomb)

Here we study the construction of quantum invariants with making necessary quantum corrections to the corresponding classical invariant for the same potential already obtained in classical integrable systems. Here we consider the potentials which are separable in addition. Consider the potential, $V(x,y) = x^{-2} + y^{-2}$. The above potential satisfies the eq.(4.29) and hence quantum invariant can be obtained by computing correction term $\Delta_Q f_9$. To find this, substitute the above potential and the value of coefficients f_1 - f_5 from eqs.(2.208) to (2.212) in eq.(4.9) and (4.10), we get two expressions for $\Delta_Q f_9$ as

$$\Delta_Q f_9 = -6\epsilon_2 x^{-4}y^2 - 6\sigma_0 x^2 y^{-4} + k_1(y), \tag{4.34}$$

$$\Delta_Q f_9' = -6\epsilon_2 x^{-4} y^2 - 6\sigma_0 x^2 y^{-4} + k_2(x), \tag{4.35}$$

where $k_1(y)$ and $k_2(x)$ are integration constants, which are being utilized to find the unique solution for $\Delta_Q f_9$. Hence we get

$$\Delta_Q f_9 = -6\epsilon_2 x^{-4} y^2 - 6\sigma_0 x^2 y^{-4}, \tag{4.36}$$

and corresponding quantum invariant for the becomes

$$I_Q = I_C + \hbar^2[-6\epsilon_2 x^{-4} y^2 - 6\sigma_0 x^2 y^{-4}]. \tag{4.37}$$

2. Second example (square plus Inverse square potential)

Consider the quadratic plus inverse quadratic potential, $V(x, y) = x^2 + y^2 + x^{-2} + y^{-2}$. The above potential satisfies the eq.(2.176) and hence quantum invariant can be obtained by computing correction term $\Delta_Q f_9$. To find this, substitute the above potential and the value of coefficients f_1 - f_5 from eqs.(2.208) to (2.212) in eq.(4.9) and (4.10),

we get two expression for $\Delta_Q f_9$ as

$$\Delta_Q f_9 = -6\epsilon_0 x^{-4} y^4 - -6\epsilon_0 x^4 y^{-4} + k_1'(y), \tag{4.38}$$

$$\Delta_Q f_9' = -6\epsilon_0 x^{-4} y^4 - -6\epsilon_0 x^4 y^{-4} + k_2'(x), \tag{4.39}$$

where $k_1'(y)$ and $k_2'(x)$ are integration constants, which are being utilized to find the unique solution for $\Delta_Q f_9$. Hence we get

$$\Delta_Q f_9 = -6\epsilon_0 x^{-4} y^4 - 6\epsilon_0 x^4 y^{-4}, \tag{4.40}$$

and corresponding quantum invariant for the becomes

$$I_Q = I_C + \hbar^2[-6\epsilon_0 x^{-4} y^4 - 6\epsilon_0 x^4 y^{-4}]. \tag{4.41}$$

In this above section we have studied the connection between classical integrability and quantum integrability. For quantum operators we used c-number representative. For the two degrees of freedom system that are studied in this chapter, we found the following types of relationships between the classical invariant and c-number representative of the quantum operator:

(1) They are identical in ordering rules,

(2) $O(\hbar^2)$ corrections are needed for quantum case.

It is worth to note that the classical integrable systems are very few in number and may be even fewer for quantum solvable cases. The results obtained may be useful in the study of quantum aspects of Hamiltonian systems. In higher order calculation, the construction of invariants for systems with two dimensions becomes a tedious exercise because of the elevated number of algebraic equations to be manipulated.

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1. Poster presentation on, *"Two-dimensional Complex Invariants for Non-Hermitian PT -symmetric Hamiltonian Systems"* at **International Conference on Statistical Physics and Nonlinear Dynamics**, (CSPND-2012) from Mar 12-16, 2012, organized by S. N. Bose Center for Basic Science, Kolkata, India.

2. Presented paper on, *"Two-dimensional Complex Invariants for PT-symmetric Hamiltonian Systems"* at **International Conference on Mathematical Modeling and Applied Soft Computing**, (MMASC-2012) from July 11-13, 2012, organized by Coimbatore Institute of Technology, Coimbatore, India.

3. Poster presentation on, *"Integrals For Time-dependent Complex Dynamical System In One Dimension"* at **Chandigarh Science Congress** (CNSC-2011) from Feb 26-28, 2011, organized by Department of Physics, Panjab University, Chandigarh, India.