Solitons in optics

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Chapter 1

Introduction

Optical solitary waves, temporal and spatial solitons, have been the subject of intense theoretical and experimental studies in recent years. Solitons (localized pulses in time or bounded self-guided beams in space) evolve from a nonlinear change in the refractive index of a material induced by the light intensity distribution. When the combined effects of the refractive nonlinearity and the pulse dispersion (in the case of temporal solitons) or beam diffraction (in the case of spatial solitons) exactly compensate each other, the pulse or beam propagates without change in shape and is said to be self-trapped. Nonlinear effects responsible for soliton formation in optical fibers are, in general, weak and Kerr-like, i.e. they induce a local index change directly proportional to the light intensity. In this case the main nonlinear equation governing the pulse evolution is the famous cubic nonlinear Schrödinger equation (NLSE) for the complex amplitude envelope of the electric field which, depending on the sign of the group-velocity dispersion, has two distinct types of localized solutions, bright or dark solitons. These two types of waves look like two members of a general family of localized solutions. However, these two types of solitary waves have completely different nature and result from quite different physics.

1.1 Solitary waves and solitons

A solitary wave is a non-singular and localized wave which propagates without change of its properties (shape, velocity etc.). It arises due to delicate balance between nonlinear and dispersion effects of a medium. It was first observed in 1834 by John Scott Russell while he was conducting experiments to determine the most efficient design for canal boats. From his experiments, he made two key discoveries [1]
1. The existence of solitary waves which are long and shallow water waves of permanent form.

2. The speed of propagation, \(v\), of solitary wave in a channel is given by

\[
v = \sqrt{g(h + \eta)},
\]

where \(\eta\) is the amplitude of wave, \(h\) is the depth of channel and \(g\) is the force due to gravity.

This phenomenon attracted some attention of scientists at that time but the theoretical basis for this phenomenon was given by two Dutch physicists, Korteweg and de Vries in 1898, who presented their famous equation (known as KdV equation) for the evolution of long waves in a shallow one-dimensional water channel which admits solitary wave solution. In 1965, Zabusky and Kruskal [2] solved the KdV equation numerically as a model for nonlinear lattice and found that solitary wave solutions interacted elastically with each other. Due to this particle-like property, they termed these solutions as solitons. Thus, a soliton is a self-reinforcing solitary wave solution of a NLEE which

- represents a wave of permanent form.
- is localized, so that it decays or approaches a constant value at infinity.
- is stable against mutual collisions with other solitons and retains its identity.

Solitons have various other interesting features, some of which are given below:

### 1.1.1 Nonlinear superposition

In linear systems, there is simple way to make new solutions from the known ones i.e. with the help of superposition principle. Before the discovery of solitons, there was no analogue of this principle for nonlinear equations, but the way that a 2-soliton solution can be viewed as a combination (though not a simple linear combination) of two 1-soliton solutions leads to a recognition that (at least for soliton equations) that there is a nonlinear superposition principle as well. Physically, when two solitons of different amplitudes (and hence of different speeds) are placed far apart (Figure 1) the taller (faster) wave on left of the shorter (slower) wave, the taller one eventually catches up to the shorter one and then overtakes it. When this happens, they undergo a nonlinear interaction and emerge from interaction completely preserved in form and speed with only a phase shift.
1.1.2 Integrability

Before the discovery of solitons, mathematicians were under the impression that one could not solve nonlinear partial differential equations exactly. However, solitons led us to recognize that through a combination of such diverse subjects such as quantum physics and algebraic geometry, we can actually solve some nonlinear equations exactly, which gives us a tremendous “window” into what is possible in nonlinearity.

1.1.3 Particle-like behavior

The particle-like behavior of solitons (that they are localized and preserved under collisions) leads to a large number of applications. On the one hand, we hope to be able to use solitons to better understand real particles. This is already true to some extent: there are soliton models of nuclei and the technique known as bosonization allows us to view particles like electrons and positrons as being solitons in appropriate situations. However, there is so far no general well-developed quantum theory in which particles are described as solitons. Still, there are macroscopic phenomena, such as internal waves on the ocean, and the behavior of light in fiber optic cable just to name a few phenomena, which are known understood in terms of solitons.

1.2 Optical solitons

In the context of nonlinear optics, solitons are classified as being either temporal or spatial, depending on whether the confinement of light occurs in time or space during
wave propagation. Temporal solitons represent optical pulses that maintain their shape, whereas spatial solitons represent self-guided beams that remain confined in the transverse directions orthogonal to the direction of propagation. Both types of solitons evolve from a nonlinear change in the refractive index of an optical material induced by the light intensity - a phenomenon known as the *optical Kerr effect* in the field of nonlinear optics. The intensity dependence of the refractive index leads to spatial self-focusing (or self-defocusing) and temporal self-phase modulation (SPM), the two major nonlinear effects that are responsible for the formation of optical solitons. A spatial soliton is formed when the self-focusing of an optical beam balances its natural diffraction-induced spreading. In contrast, it is the SPM that counteracts the natural dispersion-induced broadening of an optical pulse and leads to the formation of a temporal soliton. In both cases, the pulse or the beam propagates through a medium without change in its shape and is said to be self-localized or self-trapped.

The earliest example of a spatial soliton corresponds to the 1964 discovery of the nonlinear phenomenon of self-trapping of continuous-wave (CW) optical beams in a bulk nonlinear medium. Self-trapping was not linked to the concept of spatial solitons immediately because of its unstable nature. During the 1980s, stable spatial solitons were observed using nonlinear media in which diffraction spreading was limited to only one transverse dimension. Fig. 1.2 shows an example of the spatial soliton formed inside a semiconductor waveguide. The beam diffracts at low input powers but nearly maintains its original shape when the peak power is adjusted to correspond to a spatial soliton.

The earliest example of a temporal soliton is related to the discovery of the nonlinear phenomenon of self-induced transparency in a resonant nonlinear medium. In this case, an optical pulse of a specific shape and energy propagates through the nonlinear medium unchanged in spite of large absorption losses. Another example of a temporal soliton was found in 1973, when it was discovered that optical pulses can propagate inside an optical fiber (a dispersive nonlinear medium) without changing their shape if they experience anomalous dispersion. Propagation of such solitons in optical fibers was observed in a 1980 experiment. Since then, fiber solitons have found practical applications in designing long-haul fiber-optic communication systems.

In 1973 it was discovered that optical fibers can support another kind of temporal solitons when the group-velocity dispersion (GVD) is "normal". Such solitons appear as intensity dips within a CW background and are called dark solitons. To make the distinction clear, standard pulse-like solitons are called bright solitons. Temporal dark
solitons attracted considerable attention during the 1980s. Spatial dark solitons can also form in optical waveguides and bulk media when the refractive index is lower in the high-intensity region (self-defocusing nonlinearity), and they have been studied extensively. During the decade of the 1990s, many other kinds of optical solitons were discovered. Examples include spatiotemporal solitons (also called light bullets), Bragg solitons, vortex solitons, vector solitons, and quadratic solitons.
Chapter 2

Nonlinear Schrödinger equation

The basic equation governing the propagation of pulses in optical fibers is known as the nonlinear Schrödinger equation (NLSE) [3], whose general form is

\[ i \frac{\partial \psi}{\partial z} + a_1 \frac{\partial^2 \psi}{\partial t^2} + a_2 |\psi|^2 \psi = 0. \] (2.1)

where \( \psi(z,t) \) is the complex envelope of the electric field, \( a_1 \) is the parameter of group velocity dispersion, \( a_2 \) represents cubic nonlinearity. As its name suggests, Eq. (2.1) is similar to the well-known Schrödinger equation of quantum mechanics. Here, of course, it has nothing to do with quantum mechanics. Rather, it is just Maxwell’s equations, adapted to field propagation in single-mode optical fiber. However, the analogy to quantum mechanics may be instructive to some, as the nonlinear term is analogous to a negative potential energy, which allows the possibility of self-trapped pulse solutions.

The solutions \( \psi(z,t) \) of Eq. (2.1) directly yield the temporal form of the pulse as seen by an observer at the location \( z \); thus, in the context of telecommunications, time and distance are typically on the scales of picoseconds and kilometers, respectively. The second term in the equation, the one involving the second derivative with respect to time, describes the effects of chromatic dispersion. It is important to note that this linear term, when acting by itself, does nothing to change the frequency spectrum of the pulse. It serves only to broaden (or narrow) the pulse in time. The third term in the equation represents the nonlinear term. Note that it is just the pulses intensity envelope times \( \psi \) itself. It is based on the fact that the index of refraction, as will be detailed shortly, is dependent on the light intensity. It is important to note that this term, when acting by itself, does nothing to change the pulse shape in time. It serves only to broaden (or narrow) the pulse in the frequency domain.
2.1 Higher order effects

The propagation equation (2.1) has been successful in explaining the dynamics of picosecond pulses. But, during the past several years, ultrashort (femtosecond) pulses have been extensively studied due to their wide applications in many different areas, like ultrahigh-bit-rate optical communication systems, ultrafast physical processes, infrared time-resolved spectroscopy, optical sampling systems, etc. [4]. To produce ultrashort pulses, the intensity of the incident light field increases which leads to non-Kerr nonlinearities, changing the physical feature of the system. The dynamics of such systems should be described by the NLSE with higher order terms, such as the third-order dispersion, self-steepening and self-frequency shift [5, 6]. Moreover, in some physical situations cubic-quintic nonlinear terms arises [7, 8], due to non-Kerr nonlinearities, from a nonlinear correction to the refractive index of a medium. The higher order NLSE can be written as

\[ i\psi_t + a_1\psi_{tt} + a_2|\psi|^2\psi + a_3|\psi|^4\psi + i [a_4(|\psi|^2\psi)_t + a_5\psi(|\psi|^2)_t + a_6\psi_{tt}t] = 0, \]  

(2.2)

where \( a_3 \) represent quintic nonlinearities, \( a_4 \) is the self-steepening (SS) coefficient, \( a_5 \) is the self-frequency shift (SFS) coefficient and \( a_6 \) represents the third order dispersion coefficient.

Self-steepening and self-frequency shifting can suppress pulse splitting as it shocks the trailing edge of the pulse and introduces a low frequency spectral component for a pulse in the anomalous dispersion regime [where very narrow temporal features are created due to the interplay between group velocity dispersion (GVD) and self-focusing]. In the normal dispersion regime (where narrow temporal features are not produced), SS and SFS are significant effects only for extremely short input pulses; in this case SS and SFS do not suppress pulse splitting in time or transverse spatial dimensions; rather, a shock at the trailing edge of the pulse is initially produced, spatial and temporal pulse splitting occurs, eventually enhancement of the leading pulse results, and a high frequency tail and a low frequency spectral component are generated [9]. The amplitude of the leading pulselet becomes larger than the trailing pulselet. SFS enhances the effects introduced by SS and also shifts the frequency of the pulse. The effects of SS have been well studied in one spatial dimension (i.e., optical fibers) where the transverse spatial dimensions are eliminated due to transverse mode constraints. Figure 2.1 shows the calculated pulse shapes at \( Z = 10 \) and 20 for the dispersionless case by setting \( a_1 = a_6 = 0 \) [10]. As the pulse propagates inside the fiber, it becomes asymmetric, with its peak shifting toward the trailing edge. As a result, the trailing edge becomes steeper and steeper with increasing
Figure 2.1: Self-steepening of a Gaussian pulse in the dispersionless case. Dashed curve shows the input pulse shape at $z = 0$. 

![Graph showing self-steepening of a Gaussian pulse](image)
Chapter 3

Soliton solutions for higher order NLSE

We have obtained soliton solutions for the Eq. (2.2) when $a_6 = 0$. The effect of third-order dispersion is significant for femtosecond pulses when the GVD is close to zero. However, it can be neglected for the pulses whose width is of the order of 100 fs or more, having power of the order of 1 W and GVD far away from zero [9]. But, the effect of self-steepening as well as self-frequency shift terms are still dominant and should be retained.

To start, we have chosen the following form for the complex envelope traveling wave solutions

$$\psi(z, t) = \rho(\xi) \, e^{i[\chi(\xi) - kz]};$$

where $\xi = (t - uz)$ is the travelling coordinate, and $\rho$ and $\chi$ are real functions of $\xi$. Here, $u = 1/v$ with $v$ being the group velocity of the wave packet. Now, substituting Eq. (3.1) in Eq. (2.2) and separating out the real and imaginary parts of the equation, we arrive at the following coupled equations in $\rho$ and $\chi$,

$$k\rho + u\chi'\rho - a_4 \chi'' \rho + a_1 \rho'' - a_4 \chi' \rho^3 + a_2 \rho^3 + a_3 \rho^5 = 0$$

(3.2)

and

$$-u\rho' + a_1 \chi'' \rho + 2a_4 \chi' \rho' + (3a_4 + 2a_5) \rho^2 \rho' = 0.$$ (3.3)

To solve these coupled equations, we have chosen the ansatz

$$\chi'(\xi) = \alpha \rho^2 + \beta.$$ (3.4)

Using this ansatz in Eq. (3.3), we get the following relations

$$\alpha = -\frac{(3a_4 + 2a_5)}{4a_1} \quad \text{and} \quad \beta = \frac{u}{2a_1}.$$ (3.5)
Hence, the values of $\alpha$ and $\beta$ depend on the various model coefficients. It means the phase of the solution can be controlled by varying these coefficients. Now using Eqs. (3.4) and (3.5) in Eq. (3.2), we obtain

$$\rho'' + b_1 \rho^5 + b_2 \rho^3 + b_3 \rho = 0,$$

(3.6)

where $b_1 = \frac{1}{16a_1^2} [16a_1a_3 - (2a_5 + 3a_4)(2a_5 - a_4)],$ $b_2 = \frac{1}{2a_1^4} (2a_1a_2 - ua_4)$ and $b_3 = \frac{1}{4a_1} (4ka_1 + u^2).

It is interesting to note that, if $b_1 = 0$ i.e. quintic term is related to self-steepening and self-frequency shift terms, then Eq. (3.6) reduced to cubic nonlinear equation which admits dark and bright solitons.

**Case I.** $b_1 = 0$

(a) For $b_2 < 0$ and $b_3 > 0$, which implies $u > \frac{2a_1a_2}{a_4}$ and $k > \frac{-u^2}{4a_1}$, one obtains a dark soliton solution of Eq. (3.6) of the form

$$\rho(\xi) = \sqrt{-\frac{b_3}{b_2}} \tanh \left( \sqrt{\frac{b_3}{2}} \xi \right).$$

(3.7)

(b) For $b_2 > 0$ and $b_3 < 0$, which implies $u < \frac{2a_1a_2}{a_4}$ and $k < \frac{-u^2}{4a_1}$, one can find a bright soliton solution of the form

$$\rho(\xi) = \sqrt{-\frac{2b_3}{b_2}} \sech \left( \sqrt{-b_3} \xi \right).$$

(3.8)

Hence, when $b_1 = 0$ i.e. $16a_1a_3 = (2a_5 + 3a_4)(2a_5 - a_4)$, the amplitude profile will be the same as for the NLSE. However, unlike in the NLSE, both dark and bright solitons exist in the normal as well as anomalous dispersion regimes. But, both soliton solutions have mutually exclusive velocity space. The amplitude profile of typical dark and bright soliton is shown in Fig. 3.1, using the same values for model parameters as in Ref. [10], i.e. for $a_1 = 1.6001$, $a_2 = -2.6885$, $a_4 = 0.30814$ and $a_5 = 0.76604$.

**Case II.** $b_2 = 0$

Now, substituting $\rho^2 = y$ in Eq. (3.6), it can be reduced to

$$y'' + \frac{8}{3} b_1 y^3 + 4b_3 y + c_0 = 0.$$  

(3.9)

This equation can be solved for travelling wave solutions by using a fractional transformation [11]

$$y(\xi) = \frac{A + B f^2(\xi)}{1 + D f^2(\xi)},$$

(3.10)
which maps the solutions of Eq. (3.9) to the elliptic equation: $f'' \pm af \pm bf^3 = 0$, where $a$ and $b$ are real.

Our main aim is to study the localized solutions, we consider the case where $f = \text{cn}(\xi, m)$ with modulus parameter $m = 1$, which reduces $\text{cn}(\xi)$ to $\text{sech}(\xi)$. We can see that Eq. (3.10) connects $y(\xi)$ to the elliptic equation, provided $AD \neq B$, and the following conditions should be satisfied for the localized solution

$$12b_3A + 8b_1A^3 + 3c_0 = 0, \quad (3.11)$$

$$8b_3AD + 4b_3B + 4(B - AD) + 8b_1A^2B + 3c_0D = 0, \quad (3.12)$$

$$4b_3AD^2 + 8b_3BD + 4(AD - B)D + 6(AD - B) + 8b_1AB^2 + 3c_0D^2 = 0, \quad (3.13)$$

$$12b_3BD^2 + 6(B - AD)D + 8b_1B^3 + 3c_0D^3 = 0. \quad (3.14)$$

From Eq. (3.12), we find that $D = \Gamma B$, where $\Gamma = \frac{4 + 4b_3 + 8b_1A^2}{4A - 8b_3A - 3c_0}$. Using this in Eq. (3.13), we determine $B$ as $B = \frac{8b_1A + 4b_3A^2 + 4b_3A + 4(1 - A) + 3c_0A^2}{8b_1A + 4b_3A^2 + 4b_3A + 4(1 - A) + 3c_0A^2}$. By substituting these expressions in Eqs. (3.11) and (3.14), we can determine $A$ and $c_0$ for any given values of $b_1$ and $b_3$.

So, the localized solution are of the form

$$y(\xi) = \frac{A + B \text{ sech}^2(\xi)}{1 + D \text{ sech}^2(\xi)}. \quad (3.15)$$

And, $\rho(\xi)$ can be written as

$$\rho(\xi) = \sqrt{\frac{A + B \text{ sech}^2(\xi)}{1 + D \text{ sech}^2(\xi)}}. \quad (3.16)$$
The typical profile for amplitude is shown in Fig. 3.2, for \( a_1 = 1.6001, a_2 = -2.6885, a_3 = 0.0260, a_4 = 0.30814 \) and \( a_5 = 0.76604 \), and \( k = 0 \). To make \( b_2 = 0 \), we have chosen \( u \) as \( u = -27.9215 \).

Case III. \( b_3 = 0 \)

Figure 3.2: Typical amplitude profile for soliton solution given by Eq. (3.16), for values mentioned in the text.

For \( b_2 < 0 \) and \( b_1 > 0 \), the solution of Eq. (3.6) is of the following form:

\[
\rho(\xi) = \frac{1}{\sqrt{M + N\xi^2}},
\]

(3.17)

where \( M = -\frac{2b_1}{3b_2}, N = -\frac{b_2}{2} \). For this case, the typical profile for amplitude is shown in Fig. 3.3, for \( a_1 = 1.6001, a_2 = -2.6885, a_3 = 0.2174, a_4 = 0.30814, a_5 = 0.76604 \), and \( u = 4.1185 \). For \( b_3 = 0 \), \( k \) can be chosen as \( k = -121.8064 \).
Figure 3.3: Typical amplitude profile for soliton solution given by Eq. (3.17), for values mentioned in the text.
Chapter 4

Discussion

We have demonstrated that the competing cubic-quintic nonlinearity induces propagating soliton-like (dark/bright solitons) solutions in the nonlinear Schrödinger equation with self-steepening and self-frequency shift. Parameter domains are delineated in which these optical pulses exist. Also, fractional transform solitons are explored for this model. Moreover, these type of solutions, in which phase is nonlinear function of time, are known as chirped soliton solutions. Chirped pulses because of their application in pulse compression or amplification, and thus are particularly useful in the design of fiber optic amplifiers, optical pulse compressors and solitary wave based communications links [12, 13].
Bibliography