STUDY OF SOLITARY WAVE SOLUTIONS FOR A CLASS OF NON LINEAR EQUATIONS

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This thesis is dedicated to my family,
friends & teachers
for their love, endless support
and encouragement.

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Abstract

Nonlinear phenomena have been observed in many areas of physics such as plasma physics, optical communication, solid state physics, hydrodynamics, Bose Einstein condensation etc. These phenomena are described by nonlinear evolution equations (NLEEs) such as the nonlinear Schrödinger equation (NLSE), the Korteweg de Vries equation (KdV), sine Gordon equation and many others. The study of the exact solutions like solitary wave and periodic solutions to these NLEEs plays an important role in explanation of various nonlinear phenomena and now they have emerged as a major tool for research. Solitary waves and solitons have widespread applications in a large number of physical, chemical and biological systems. In this thesis, a systematic analysis of solitary wave solutions has been done for a class of nonlinear evolution equations (NLEEs) and their relevance has been studied for various physical situations.

Various methods have been used to find solitary wave solutions of nonlinear equations. Ansatz method has been used to solve higher order nonlinear Schrödinger equation (HNLSE) with cubic-quintic nonlinearity and soliton (dark/bright and double kink type) solutions have been obtained. Factorization method has been extended to solve a class of driven NLEEs and topological stable, particular as well as general solutions have been obtained. The role of nonlinear waves in the energy transfer in biological processes through DNA molecule via solitonic solutions have been studied and the solution so obtained has been generalized using Riccati parameter method.

Also, most of the real physical and biological systems are inhomogeneous due to fluctuations in environmental conditions and non-uniform media. Hence the nonlinear evolution equations describing them possess time varying or space varying coefficients. A system of coupled reaction diffusion equation with variable coefficients has been studied and exact solitary wave solutions have been obtained.

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Chapter 1

Introduction

In the last few decades, nonlinear science has evolved as a dynamic tool to study the mysteries of complex natural phenomena. In general, nonlinear science is not a new subject or branch of science, although it delivers significantly a new set of concepts and remarkable results. Unlike quantum physics and relativity, it encompasses systems of different scales and objects moving with any velocity. Hence, due to feasibility of nonlinear science on every scale, it is possible to study same nonlinear phenomena in very distinct systems with the corresponding experimental tools. The general theme underlying the study of nonlinear systems is nonlinearity present in the system.

Nonlinearity is exciting characteristic of nature which plays an important role in dynamics of various physical phenomena [1, 2] such as in nonlinear mechanical vibrations, population dynamics, electronic circuits, laser physics, astrophysics (e.g planetary motions), heart beat, nonlinear diffusion, plasma physics, chemical reactions in solutions, nonlinear wave motions, time-delay processes etc. A system is called nonlinear if its output is not proportional to its input. For example, a dielectric material behaves nonlinearly if the output field intensity is no longer proportional to the input field intensity. For most of the real systems, nonlinearity is more regular feature as compared to the linearity. In general, all natural and social systems behave nonlinearly if the input is large enough. For example, the behavior of a spring and a simple pendulum is linear for small displacements. But for large displacements both of them act as nonlinear systems. A system has a very different dynamics mechanism in its linear and nonlinear regimes.

Most of the nonlinear phenomena are modelled by nonlinear evolution equations (NLEEs) having complex structures due to linear and nonlinear effects. The advancement of high-speed computers and new techniques in mathematical softwares and analytical methods to study NLEEs with experimental support has stimulated the theoretical and experimental research in this area. The investigation of the exact solutions, like solitary wave and periodic, of NLEEs plays a vital role in the description of nonlinear physical phenomena. The wave propagation in fluid dynamics, plasma, optical and elastic media are generally modelled by bell-shaped and kink-shaped solitary wave solutions. The existence of exact solutions, if available, to NLEEs helps in verification of numerical analysis and is useful in the study of stability analysis of solutions. Moreover, the search of exact solitary wave solutions led to the discovery of new concepts, such as solitons, rogue waves, vortices, dispersion-managed solitons, similaritons, supercontinuum generation, modulation instability, complete integrability, etc.

1.1 Solitary waves and solitons

Much attention has been given to the study of solitary waves and solitons during the last few years because of their widespread applications in various fields. A solitary wave is a non-singular, localized and dispersionless solution of nonlinear evolution equations which continues to travel with a constant velocity without dissipating its energy. These waves arise due to a delicate balance between nonlinear and dispersive effects of a medium. Those solitary waves which are stable against mutual collisions and retain their identity after collisions and continue to travel with their original shapes and speeds are called solitons. The term ‘soliton’ was coined by the American mathematicians Norman Zabusky and Martin Kruskal [3] to describe the particle like behavior of solitary waves. Because of their remarkable stability, solitons are capable of propagating long distance without attenuation.

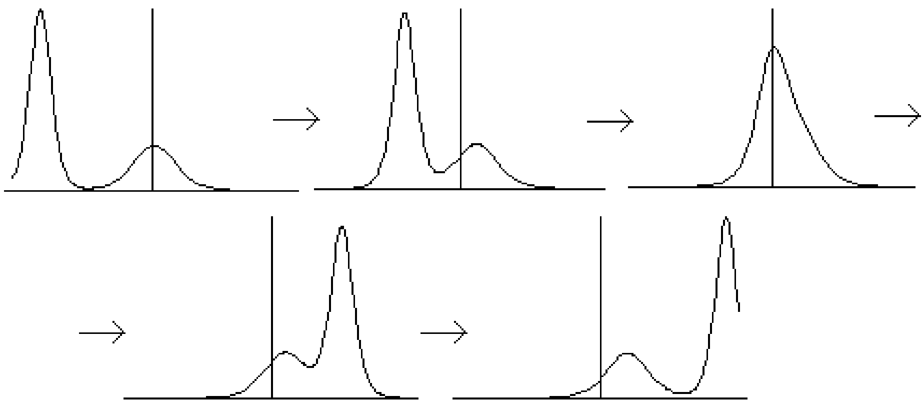

Figure 1.1: Collision of two solitons

Fig. 1.1 represents the collision of two solitons of different amplitudes (and hence of different speeds). The taller (faster) wave on the left catches upto the shorter (slower) wave and then overtakes it. These solitons undergo a non linear interaction and emerge from the interaction completely preserved in form and speed with only a phase shift. Due to this property, solitons have been used as a constructive element to formulate the complex dynamical behavior of wave systems throughout science - from hydrodynamics to nonlinear optics, from plasmas to waves in condensed matter physics, from nuclear physics to particle physics, from biophysics to astrophysics.

The first observation of solitary wave was made in 1834 by the Scottish scientist and engineer John Scott Russel (1808-1882) [4], while he was conducting experiments to determine the most efficient design for canal boats. There was subsequently a gap of sixty years between Scott Russel's observation of shallow water solitary waves and any theoretical treatment of the phenomenon. The initial theoretical confirmation of Russel's work had to wait until 1895 when two Dutch scientists D. J. Korteweg and G de Vries [5] formulated a mathematical equation to provide an explanation of the phenomenon observed by Russel. They derived the famous KdV equation to describe the propagation of surface waves in water. In 1965, Zabusky and Kruskal [3] solved the KdV equation numerically as a model for nonlinear lattice and found that solitary wave solutions interacted elastically with each other. Due to this particle-like property, they termed these solitary wave solutions as solitons.

After that, the concept of soliton was accepted in general and soon Gardner *et al.* in 1967 reported the existence of multi-soliton solutions of KdV equation using inverse scattering method [6]. One year later, Lax generalized these results and proposed the concept of Lax pair [7]. Zakharov and Shabat [8] used the same method to obtain exact solutions for nonlinear Schrödinger equation. Hirota [9] introduced a new method, known as Hirota direct method, to solve the KdV equation for exact solutions for multiple collision of solitons. In 1974, Ablowitz *et al.* [10] showed that inverse scattering method is analog of the Fourier transform and used it to solve a wide range of new equations, such as the modified KdV equation, the nonlinear Schrödinger equation and the classical sine-Gordon equation. These techniques stimulated the study of soliton theory in various fields such as optical communications, molecular biology, chemical reactions, oceanography etc.

1.2 Nonlinear evolution equations (NLEEs) and their examples

Nonlinear evolution equations (NLEEs) arise throughout the nonlinear science as a dynamical description (both in time and space dimensions) to the nonlinear systems. NLEEs are very useful to describe various nonlinear phenomena of physics [11, 12], chemistry, biology and ecology, such as fluid dynamics, wave propagation, population dynamics, nonlinear dispersion, pattern formation etc. Hence, NLEEs evolved as a useful tool to investigate the natural phenomena of science and technology. A NLEE is represented by a nonlinear partial differential equation (PDE) which contains a dependent variable (the unknown function) and its partial derivatives with respect to the independent variables. During the last few years several new nonlinear evolution equations have been formulated depending on the physical phenomenon to be described and their soliton solutions which have physical relevance are being analyzed.

There are many non linear partial differential equations which are found to be completely integrable and admit solitonic solutions. For example the Korteweg-de-Vries (KdV) eqn, modified KdV (mKdV) equation, the nonlinear Schrödinger equation (NLSE), higher order nonlinear Schrödinger equation (HNLSE), the sine-Gordon (SG) equation, Burger Fisher equation, system of reaction diffusion equations etc. Some of these are discussed below.

1.2.1 KdV equation

KdV equation is the simplest wave model that combines dispersion with non linearity. The KdV type equations have been the most important class of NLEEs with numerous applications in physical sciences and engineering. It was first of all investigated theoretically in 1872 by Lord Rayleigh [13] and French applied mathematician Joseph Boussinesq [14] as a model to surface water waves. A simple generalization of KdV equation is

$$u_t + uu_x + u_{xxx} = 0, \quad -\infty < x < \infty, 0 \leq t < \infty, \tag{1.1}$$

where u is the wave amplitude; x, t are space and time variables respectively and subscripts represent differentiation w.r.t. the relevant variable. It is a nonlinear, dispersive partial differential equation of third order. In this equation, the first term is the time evolution term, second term is the nonlinear term and the third term introduces dispersion. A travelling wave solution of permanent form occurs due to a balance between the dispersive term and the nonlinear term.

This equation is considered as the first integrable nonlinear partial differential equation for solitary waves. Based on his personal observations of solitary waves since 1830, Russel insisted that such waves do occur. Korteweg and de-Vries [5] proved Russel was correct by finding explicit, closed form, travelling wave solutions to their equation that do not decay rapidly and so represent a solitary wave. KdV equation can be applied to understand the properties of many physical systems which are weakly nonlinear and weakly dispersive. For example [15] shallow water waves with weakly nonlinear restoring forces, ion acoustic waves in a plasma, acoustic waves in a crystal lattice, internal waves in oceans, blood pressure waves etc.

1.2.2 Sine Gordon (SG) equation

This equation is similar to 'Klein-Gordon equation' in relativistic field theories. This equation is a relativistic nonlinear equation which possesses soliton solutions. The structure of soliton solutions of sine-Gordon equation is similar to that of KdV and mKdV equations. It exhibits kink, antikink and breather type solutions in one dimension.

The typical type of SG equation is given by

$$u_{tt} \ - \ u_{xx} \ + \ sin\ (u) \ = \ 0, \tag{1.2}$$

where u is the real scalar field.

It was first studied in the course of investigation of surfaces of constant Gaussian curvature (also called pseudospherical surfaces). It has a wide range of applications in physics, not only in relativistic field theories but in solid state physics, mechanical transmission lines, crystal dislocation, shape waves, Josephson junction, nonlinear optics etc.

1.2.3 Nonlinear Schrödinger equation (NLSE)

Nonlinear Schrödinger equation (NLSE) is one of the fundamental dynamic models in nonlinearity. It represents the propagation of a wave through a medium with both nonlinear and dispersive responses. In its simplest form it can be expressed as

$$\iota\psi_t \ + \ \psi_{xx} \ + \ g\ |\psi|^2\psi \ = \ 0, \quad -\infty < x < \infty,\ t > 0, \tag{1.3}$$

where ψ is the complex field function and g is a constant.

The term involving second order derivative in the direction of propagation describes the dispersion and the third term describes the nonlinearity of the system.

The dispersion effect makes the waveform spread and the nonlinear effect causes the steepening of the waveform. Due to these competing effects, a stationary wave called the solitary wave exists.

NLSE has a class of exact localized solutions which have applications in hydrodynamics, nonlinear optics, heat pulses in solids, plasmas, electromagnetism and various other instability phenomena. It also features predominantly in the problem of optical pulse propagation in asymmetric, twin core optical fibers [16–18].

NLSE as given above does not give correct prediction for pulse widths smaller than one picosecond. For example, in solid state solitary lasers, where pulses as short as 10 femtoseconds are generated, the NLSE breaks down. For transmitting pulses at a higher bit rate, it is necessary to propagate ultra short pulses. For such short pulses, higher order effects such as third order dispersion, self steepening, Raman Scattering etc. need to be considered. For this, there are other variants of NLSE which include externally driven NLSE, higher order nonlinear Schrödinger equation (HNLSE) which feature predominantly in propagation of optical pulses.

1.2.4 Nonlinear reaction diffusion equation

Reaction diffusion equations are the mathematical models of those physical or biological systems in which the concentration of one or more substances distributed in space varies under the effect of two processes: reaction and diffusion. Nonlinear reaction diffusion equations (NLRD), with convective term or without it, have attracted considerable attention, as it can be used to model the evolution systems in real world. The general form of NLRD type equations is

$$u_t + vu^m u_x = Du_{xx} + \alpha u - \beta u^n, \tag{1.4}$$

where $u(x,t)$ represents the concentration or density of substance, D is diffusion coefficient, v is the convection coefficient, α, β are reaction term coefficients and m, n are real numbers. NLRD equations play an important role in the qualitative description of many phenomena such as flow in porous media, heat conduction in plasma, chemical reactions, population genetics, image processing and liquid evaporation.

1.3 Inhomogeneous nonlinear evolution equations

In the study of NLEEs as a model to an actual physical system, it is generally essential to consider the factors causing deviation from the actual system, originating due to the dissipation, environmental fluctuations, spatial modulations and other forces. In order to consider some or all of these factors, it is necessary to add the appropriate perturbing terms in NLEEs. Hence, inhomogeneous NLEEs are more realistic to study the dynamics of real physical systems.

1.3.1 NLEEs with variable coefficients

The physical phenomena in which NLEEs with constant coefficients arise, tend to be highly idealized. But for most of the real systems, the media may be inhomogeneous and the boundaries may be nonuniform, e.g. in plasmas, superconductors, optical fiber communications, blood vessels and Bose-Einstein condensates. Therefore, the NLEEs with variable coefficients are supposed to be more realistic than their constant-coefficient counterparts in describing a large variety of real nonlinear physical systems. Some phenomena which are governed by variable coefficient NLEEs, are given as

- In a real optical fiber transmission system, there always exist some non-uniformities due to the diverse factors that include the variation in the lattice parameters of the fiber media and fluctuations of the fiber's diameter. Therefore, in real optical fiber, the transmission of soliton is described by the NLSE with variable coefficients [19]. Sometimes, the optical waveguide is tapered along the waveguide axis to improve the coupling efficiency between fibers and waveguides in order to reduce the reflection losses and mode mismatch. The propagation of beam through tapered graded-index nonlinear waveguides is governed by the inhomogeneous NLSE [20].

- The evolution equation for the propagation of weakly nonlinear waves in shallow water channels of variable depth and width and also in plasmas having inhomogeneous properties of media, is obtained as a variable coefficient KdV equation [21, 22].

- The dynamics of the matter wave solitons in Bose-Einstein condensates (BEC) can be controlled through artificially inducing the inhomogeneities in system [23]. The dynamical behavior of inhomogeneous BEC is described by the variable coefficient Gross-Pitaevskii equation also known as generalized nonlinear Schrödinger equation (GNLSE).

- Generally, the NLRD systems are inhomogeneous due to fluctuations in environmental conditions and nonuniform media. It makes the relevant parameters space or time dependent because external factors make the density and/or temperature change in space or time [24, 25].

- To describe waves in an energetically open system with a monotonically varying external field, sine-Gordon equation with dissipation and variable coefficient on the nonlinear term is considered [26].

1.3.2 NLEEs in the presence of external source

To control the dynamics of a nonlinear system, it is essential to investigate the effects of dissipation, noise and external force on the system. Dissipation leads to loss of energy and hence affects the dynamics of system under consideration, whereas the external tunable driving force acts as a source of energy and helps in stabilizing the dynamical system. Barashenkov *et al.* [27] considered the parametrically driven damped NLSE and showed the existence of stable solitons only if the strength of the driving force would be more than the damping constant. Recently, work on forced NLSE has attracted much attention [28–31], as it arises in many physical problems, such as the plasmas driven by rf-fields, pulse propagation in twin-core fibers, charge density waves with external electric field, double-layer quantum Hall (pseudo) ferromagnets, etc.[28].

1.4 Methodology

Although NLEEs play a significant role in explanation of complex nonlinear phenomena, it was a difficult task to find their solutions because the mathematical models for describing NLEEs were not available. But later on, many methods were devised to obtain exact solutions of NLEEs. These days a large number of methods are available for constructing exact solutions of nonlinear partial differential equations. For example, quadrature method, ansatz method, method of separation of variables, singularity analysis (Painlevé method) for partial differential equations (PDE's), Hirota method, Inverse scattering method, the functional variable method, Factorization method etc. But these methods are problem dependent ; that is some methods work well with a certain problem but not with others. In this section, we will briefly outline some of the above mentioned methods.

1.4.1 Method of quadratures

This method is used for finding solutions of (1+1) dimensional field theories [32]. The Lagrangian in field theory is given by

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi), \tag{1.5}$$

where$\mu = 0, 1$; represent the space time coordinates.

The Euler Lagrange equation is given by

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)} \right) - \frac{\partial L}{\partial \phi} = 0. \tag{1.6}$$

Substituting the lagrangian in this equation and considering the static solutions only, we have

$$\frac{d^2\phi}{dx^2} = \frac{dV}{d\phi}. \tag{1.7}$$

Multiplying this equation by $\frac{d\phi}{dx}$ and integrating w.r.t. x, we have

$$\left(\frac{d\phi}{dx} \right)^2 = 2V(\phi), \tag{1.8}$$

which on further integration gives

$$\int \frac{d\phi}{\sqrt{2V(\phi)}} = x - x_0, \tag{1.9}$$

which implies $g(\phi) = x - x_0$. If $g(\phi)$ can be inverted, explicit solution can be obtained. If $V(\phi)$ has two minima, it will lead to kink solution and if it has a single minimum, then non topological solution will be obtained.

1.4.2 Ansatz method

There are various methods like algebraic method [33], tanh method [34], exponential function method [35] etc. in which non linear partial differential equations can be solved by choosing a suitable ansatz. In this method, we chose a trial function as a solution of the equation to be solved in the form of a steady state wave function and substitute this ansatz in the given equation. Then we collect the coefficients of different powers of trial function and equate them to zero to find the unknown parameters. Here, in our work, we have used this method to obtain solutions of HNLSE for a range of parameters.

1.4.3 Painlevé analysis

The Painlevé test is used as a means of predicting whether an equation is likely to be integrable or not. The Painlevé property states that an ordinary differential equation (ODE) is integrable if the equation admits no moving singularities other than simple poles. This approach was later extended by Weiss, Tabor and Carnevale [36] to study the integrability properties of partial differential equations (PDE's). In this approach a PDE is considered to be integrable if its solutions are single valued about movable 'singularity manifolds':

$$\phi(z_1, z_2, ...z_n) \quad = \quad 0, \tag{1.10}$$

where ϕ is an arbitrary(movable) analytic function. In other words, a solution $x(z_i)$ to the PDE in question should have Laurent like expansion about the movable singularity manifold.

$$x(z_i) \;=\; [\phi(z_i)]^\rho \sum [\alpha_j(z_i)][\phi(z_i)]^j, \tag{1.11}$$

where ρ is a negative integer. After the substitution of x and y in the concerned differential equations, we make the leading order analysis and then determine the resonance positions. We then expand out to the highest resonance and check the integrability of the system.

1.4.4 Factorization method

In 1940, factorization method was developed by Schrödinger to solve Schrödinger equation. Almost after a decade in 1951, Infeld and Hull [37] developed a different type of factorization technique that became widely known. In 1984, Mielnik [38] introduced factorization of quantum harmonic oscillator and since then this technique has been extended to any solvable quantum mechanical potential.

Recently Rosu and his collaborators [39, 40] developed a factorization technique to solve second order nonlinear evolution equations of the type

$$\ddot{u} + g(u)\dot{u} + F(u) = 0, \tag{1.12}$$

where the dot refers to derivative w.r.t. time ($D = \frac{d}{dt}$); $g(u)$ and $F(u)$ are arbitrary functions of u.

Factorizing above equation in the form

$$[D - \phi_2(u)][D - \phi_1(u)]u = 0, \tag{1.13}$$

where ϕ_1 and ϕ_2 are factorizing functions to be determined later from the consistency conditions.

Eq. (1.13) implies

$$\ddot{u} - \frac{d\phi_1}{du} u\dot{u} - \phi_1\dot{u} - \phi_2\dot{u} + \phi_1\phi_2 u = 0. \tag{1.14}$$

After regrouping the terms, we get

$$\ddot{u} - \left(\phi_1 + \phi_2 + \frac{d\phi_1}{du} u\right)\dot{u} + \phi_1\ \phi_2 u = 0. \tag{1.15}$$

Comparing Eq. (1.12) and (1.15), we have the following consistency conditions for $g(u)$ and $F(u)$

$$g(u) = -\left(\phi_1 + \phi_2 + \frac{d\phi_1}{du}u\right), \tag{1.16}$$

and

$$F(u) = \phi_1\phi_2 u. \tag{1.17}$$

If any nonlinear second order differential equation of the form (1.12) can be factorized as (1.13) satisfying conditions (1.16) and (1.17), then its particular solution can be easily found by solving the first order differential equation

$$[D - \phi_1(u)]u = 0. \tag{1.18}$$

A number of equations have been solved very easily by employing this method whose particular and generalized solutions are not easy to find by the already known analytical techniques [40–43].

1.4.5 Auxiliary equation method

Sirendaoreji and Jiong [44] proposed auxiliary equation method to solve nonlinear partial differential equations (PDE's) with constant coefficients. Suppose, for a given nonlinear evolution equation with independent variables x and t, and dependent variable u:

$$F(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...) = 0. \tag{1.19}$$

By using Galilean transformation, we can write Eq. (1.19) in travelling wave frame as

$$G(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, ...) = 0, \tag{1.20}$$

where $\xi = kx - \omega t$, k and ω are constants. Let us assume that the solution of Eq. (1.20) is of the following form

$$u(\xi) = \sum_{i=0}^{n} a_i z^i(\xi), \tag{1.21}$$

where a_i $(i = 0, 1, 2, ...)$ are real constants to be determined, n is a positive integer and $z(\xi)$ represents the solutions of the following auxiliary ordinary differential equation, viz.

$$z_\xi = az(\xi) + z^2(\xi), \tag{1.22}$$

where a, b, c are constants. To determine u explicitly, take the following four steps.

Step 1 Substitute Eq. (1.21) along with Eq. (1.22) into Eq. (1.20) and balance the highest order derivative terms with the highest power nonlinear terms in Eq. (1.20), to find the value of n.

Step 2 Again substitute Eq. (1.21), with the value of n found in Step 1, along with Eq. (1.22)into Eq. (1.20), collect coefficients of $z^i {z_\xi}^j (j = 0, 1;\ i = 0, 1, 2, ...)$ and then set each coefficient to zero, to get a set of over-determined partial differential equations for $a_i (i = 0, 1, 2...)$, k and ω.

Step 3 By solving the equations obtained in Step 2, get the explicit expressions for $a_i (i = 0, 1, 2...)$, k and ω.

Step 4 By using the results obtained in previous steps, obtain the exact travelling wave solution of Eq. (1.19) from Eq. (1.21) depending on the solution $z(\xi)$ of Eq. (1.22).

This method can also be used to solve nonlinear PDE's with variable coefficients by converting these equations into ODE's in accelerated travelling wave frame with the help of extended Galilean transformation [45]. Yomba [46] follow this procedure to solve KdV equation with variable coefficients using auxiliary equation method and Bekir *et al.* [47] used it to solve Zakharov-Kuznetsov equation with variable coefficients using exp. function method.

1.5 Outline of thesis

The layout of the thesis is as follows.

In Chapter 2, we demonstrate that the competing cubic-quintic nonlinearity induces propagating soliton-like (dark/bright solitons) and double kink solitons in the nonlinear Schrödinger equation with self-steepening and self-frequency shift. It is observed that the nonlinear chirp associated with each of these optical pulses is directly proportional to the intensity of the wave and saturates at some finite value as the retarded time approaches its asymptotic value. We further show that the amplitude of the chirping can be controlled by varying self-steepening term and self-frequency shift.

Chapter 3 begins with a brief introduction of factorization method. This method is then extended to solve Convective Fisher equation driven by a constant force and a particular solution is obtained. We also present Riccati generalization of driven convective Fisher equation. An implicit solution of Duffing-Van der Pol oscillator driven by constant force is also obtained using the same method.

In Chapter 4, we study the nonlinear dynamics of DNA, for longitudinal and transverse motions, in the framework of the microscopic model of Peyrard and Bishop. The coupled nonlinear partial differential equations for dynamics of DNA model have been solved for solitary wave solution which is further generalized using Riccati parameterized factorization method.

Chapter 5 shows the existence of exact solitary wave solutions, with non-trivial time dependence, for a class of nonlinear reaction-diffusion equations with time-dependent coefficients of convective term and reaction term, by using the auxiliary equation method.

In conclusion, Chapter 6 discusses the results obtained in the preceding chapters and provides a summary of the significant findings of our work.

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Chapter 2

Chirped femtosecond solitons and double kink solitons for the higher order nonlinear Schrödinger equation

2.1 Introduction

The nonlinear Schrödinger equation (NLSE) in its many versions has various applications in different fields, such as nonlinear optics [1], Bose-Einstein condensates [2], biomolecular dynamics [3], and others. NLSE describes the dynamics of picosecond pulses which propagate in nonlinear media due to the delicate balance between group velocity dispersion (GVD) and Kerr nonlinearity. Thus, the dynamics of such light pulses can be conveniently described by NLS family of equations with cubic nonlinear terms only. However, during the past several years, ultrashort (femtosecond) pulses have been extensively studied due to their wide applications in different areas, like ultrahigh-bit-rate optical communication systems, ultrafast physical processes, infrared time-resolved spectroscopy, optical sampling systems, etc. [4]. To produce ultrashort pulses, the intensity of the incident light field increases which leads to non-Kerr nonlinearities, changing the physical feature of the system. At such high intensities, the classic NLSE fails to describe the propagation of light pulses in optical fibers. The dynamics of such systems should be described by the NLSE with higher order terms, such as the third-order dispersion, self-steepening and self-frequency shift [5, 6]. Moreover, in some physical situations cubic-quintic nonlinear terms arise [7, 8] due to non-Kerr nonlinearities, from a nonlinear correction to the refractive index of a medium. In general, unlike NLSE, these models with non-Kerr effects are not completely integrable and cannot be solved exactly by the inverse scattering transform method [9]. Hence, they do not have soliton solutions; however, they do have solitary wave solutions which have often been called solitons.

The effect of third-order dispersion is significant for femtosecond pulses when the GVD is close to zero. However, it can be neglected for the pulses whose width is of the order of 100 fs or more, having power of the order of 1 W and GVD far away from zero [10]. But, the effect of self-steepening as well as self-frequency shift terms are still dominant and should be retained. Under these conditions, we have assumed the higher order NLSE with cubic-quintic nonlinearity of the following form

$$i\psi_z + a_1\psi_{tt} + a_2|\psi|^2\psi + a_3|\psi|^4\psi + ia_4(|\psi|^2\psi)_t + ia_5\psi(|\psi|^2)_t = 0, \qquad (2.1)$$

where $\psi(z,t)$ is the complex envelope of the electric field, a_1 is the parameter of GVD, a_2 and a_3 represent cubic and quintic nonlinearities respectively, a_4 is the self-steepening coefficient and a_5 is the self-frequency shift coefficient. These parameters are related to the width of the pulse. This equation was introduced by Kundu [11] in the study of integrability and its special cases arise in various physical and mechanical applications. Scalora et al [12] used the model (2.1), for $a_5 = 0$, to describe pulse propagation in a negative index material (NIM), where the sign of GVD can be positive or negative.

In Ref. [10, 13] Eq. (2.1) was solved for $a_3 = 0$ and soliton-like solutions with nonlinear chirp were obtained. In this chapter, we obtain the soliton solutions with different form of chirping. We observe that for certain parameter condition between quintic, self-steepening and self-frequency shift terms, the solutions resemble NLSE solitons with velocity selection. We also report the existence of double kink-type solitons with nonlinear chirp. In all the cases, chirping varies as directly proportional to the intensity of wave and saturates at some finite value as $t \rightarrow \pm\infty$. Further, the amplitude of chirping can be controlled by varying the self-steepening and self-frequency shift terms. It is also seen that, for same values of all parameters, the equation can have either dark/bright solitons or double kink-type solitons depending upon velocity and other parameters of wave.

2.2 NLSE and its solutions

The propagation of electromagnetic waves through an optical waveguide is governed by the famous nonlinear Schrödinger equation (NLSE). The NLSE in ideal Kerr nonlinear medium is completely integrable using the inverse scattering theory and hence all solitary wave solutions of NLSE are called as solitons. In most nonlinear systems of physical interest, we have non-Kerr or other types of nonlinearities and hence can be modelled by non-integrable generalized NLSE. The wave solutions for non-integrable equations are generally referred as optical solitary waves to distinguish them from solitons in integrable systems. But in recent literature on nonlinear optics, there is no nomenclature distinction and all solitary wave solutions in optics are referred as optical solitons.

2.2.1 Optical solitons

In nonlinear optics, the term soliton represents a light field which does not change during propagation in an optical medium due to a delicate balance between dispersive and nonlinear effects in the medium. The optical solitons have been the subject of intense theoretical and experimental studies because of their practical applications in the field of fiber-optic communications. These solitons evolve from a nonlinear change in the refractive index of a material induced by the light field. This phenomenon of change in the refractive index of a material due to an applied field is known as optical Kerr effect. The Kerr effect, i.e. the intensity dependence of the refractive index, leads to nonlinear effects responsible for soliton formation in an optical medium [14].

The optical solitons can be further classified as being spatial or temporal depending upon confinement of light in space or time during propagation. The spatial self-focusing (or self-defocusing) of optical beams and temporal self-phase modulation (SPM) of pulses are the nonlinear effects responsible for the evolution of spatial and temporal solitons in a nonlinear optical medium. When self-focusing of an optical beam exactly compensates the spreading due to diffraction, it results into the formation of spatial soliton, and a temporal soliton is formed when SPM balances the effect of dispersion-induced broadening of an optical pulse. For both soliton solutions, the wave propagates without change in its shape and is known as self-trapped. Self-trapping of a continuous-wave (CW) optical beam was first discovered in a nonlinear medium in 1964 [15]. But, these self-trapped beams were not said to be spatial solitons due to their unstable nature. First stable spatial soliton was observed in 1980 in an optical medium in which diffraction spreading was confined in only one transverse direction [16]. The first observation of the temporal soliton was linked to the nonlinear phenomenon of self-induced trapping of optical pulses in a resonant nonlinear medium [17]. Later, temporal solitons was found in an optical fiber both theoretically [18] and experimentally [19].

In standard form, the NLSE can be written as

$$\iota\psi_z \;+\; \psi_{xx} \;\pm\; g\;|\psi|^2\psi \;=\; 0, \tag{2.2}$$

where sign + (-) corresponds to self-focusing (self-defocusing) nonlinearity. In 1971, Zakharov and Shabat [20] showed that NLSE is completely integrable through the inverse scattering transform. The NLSE has a class of exact localized solutions which have applications not only in nonlinear optics but also in the fields of hydrodynamics, plasma studies, electromagnetism and many more. Here, we have given expressions of some localized solutions of NLSE which have been used further in this chapter.

Bright and dark solitons

The NLSE has bright and dark solitons depending upon the sign of nonlinearity. For self-focusing case, the NLSE admits bright solitons which decay to background state at infinity. The general form of bright soliton for NLSE is given as [20]

$$\psi(z,x) = a\ \text{sech}[a(x - vz)]\ e^{i(vx+(a^2-v^2)z/2)}, \tag{2.3}$$

where a represents the amplitude of soliton and v gives the transverse velocity of propagating soliton. The intensity expression for bright soliton will take the form

$$\begin{aligned} I_B &= |\psi(z,x)|^2 \\ &= a^2\ \text{sech}^2[a(x - vz)]. \end{aligned} \tag{2.4}$$

The evolution of bright soliton for NLSE is shown in Fig. 2.1 for typical values of a and v.

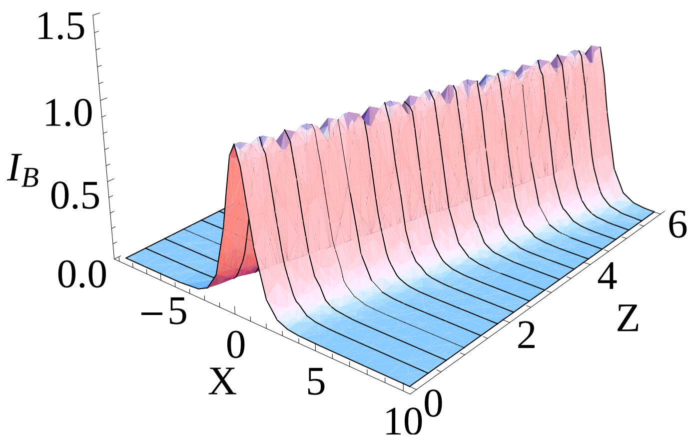

Figure 2.1: Evolution of bright soliton of the NLSE for $a = 1$ and $v = 1$.

For self-defocusing case, the NLSE has soliton solutions which do not vanish at infinity, called the dark solitons. These solitons have a nontrivial background and as the name suggests, exist in the form of intensity dips on the CW background [21].

The fundamental dark soliton for NLSE has the following general form [22]

$$\psi(z,x) = u_0[B\ \tanh(u_0B(x - Au_0z)) + iA]\ e^{-iu_0^2 z}, \tag{2.5}$$

where u_0 is CW background, and A and B satisfy the relation $A^2 + B^2 = 1$. Introducing a single parameter ϕ, we have $A = \sin\phi$ and $B = \cos\phi$ such that angle 2ϕ gives the total phase shift across the dark soliton. The intensity expression for dark soliton will take the form

$$I_D = u_0^2[1 - \cos^2\!\phi\ \text{sech}^2(u_0\cos\phi(x - u_0\sin\phi\ z))]. \tag{2.6}$$

Hence, $u_0 \sin\phi$ represents the velocity of the dark soliton and $\cos^2\phi$ gives the magnitude of the dip at the center. The evolution of a dark soliton for NLSE is shown in Fig. 2.2 (a) for typical values of u_0 and ϕ. For $\phi = 0$, the velocity of dark soliton is zero i.e. it is a stationary soliton and at the dip center intensity also drops to zero (shown in Fig. 2.2 (b)), and hence it is also called as black soliton. For other values of ϕ, the intensity of soliton does not drop to zero and these are referred as gray solitons.

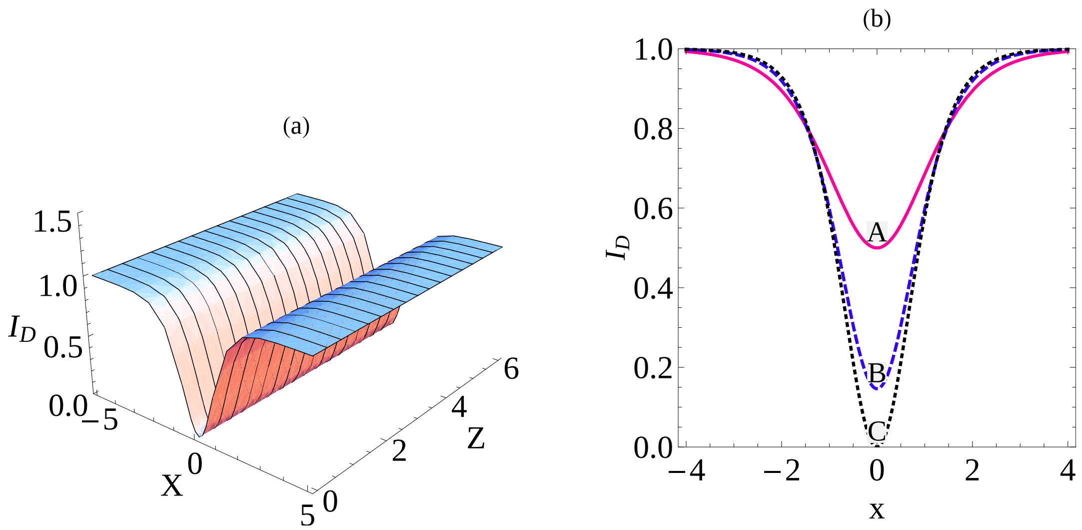

Figure 2.2: (a) Evolution of a dark soliton of the NLSE for $u_0 = 1$ and $\phi = 0$. (b) Intensity plots at $z = 0$ for different values of ϕ. Curves A, B and C correspond to $\phi = \pi/4, \pi/8$ and 0 respectively.

2.3 Chirped solitons

Nonlinear pulse propagation in optical medium is governed by nonlinear Schrödinger equation (NLSE), which is completely integrable and allows for either bright or dark solitons depending on the signs of dispersion or nonlinearity. These conventional solitons are chirp free pulses because the chirp produced by group velocity dispersion is balanced by the chirp produced by the Kerr nonlinearity [14]. However, if, in addition, one includes the gain/loss, higher order or variable coefficient terms in NLSE, then chirped solitons are possible in the optical medium [23, 24]. The chirp of an optical pulse is usually understood as the time dependence of its instantaneous frequency and can be found as the rate of change of phase of pulse. It means that if a pulse has a phase $\phi(z, t)$ then its chirp frequency is given by

$$\delta\omega(t) = -\frac{\partial\phi}{\partial t}. \tag{2.7}$$

The chirp play an important role in the subsequent pulse evolution. The chirped solitons are formed due to the growth of initially present chirp in the pulse, in contrast to the zero chirp in the case of the conventional solitons. Desaix *et al.* [25] studied the effect of linear and nonlinear chirp on the subsequent development of pulses. They have found that the properties of chirped solitons depend not only on the amplitude, but also on the form, of the initial chirp. The case of linear chirp frequency was first investigated by Hmurcik *et al.* [26] for a sech-shaped pulse with quadratic variation of phase in time. G.P. Agrawal [27] studied the pulse propagation in doped fiber amplifier and obtained soliton solutions with nonlinear chirping. Afterwards a lot of work has been done on the existence of chirped solitons in nonlinear optical systems [28–30]. The chirped solitons have been utilized to achieve efficient chirp and pedestal free pulse compression or amplification [28, 31]. Hence, chirped pulses are useful in the design of optical devices such as optical fiber amplifiers, optical pulse compressors and solitary wave based communication links [23, 32, 33]. Apart from this, chirped solitons also find applications in the generation of ultrahigh peak power pulses [34], ultrafast nonlinear spectroscopy [35] and coherent control of higher-order harmonics [36].

2.4 Chirped soliton-like solutions

We are interested to find chirped soliton-like solutions of Eq. (2.1). Hence, we have chosen the following form for the complex envelope travelling wave solutions

$$\psi(z,t) = \rho(\xi)\ e^{i[\chi(\xi)-kz]}, \tag{2.8}$$

where $\xi = (t - uz)$ is the travelling coordinate, and ρ and χ are real functions of ξ. Here, $u = 1/v$ with v being the group velocity of the wave packet. The corresponding chirp is given by $\delta\omega(t,z) = -\frac{\partial}{\partial t}(\chi(\xi) - kz) = -\chi'(\xi)$. Now, substituting Eq. (2.8) in Eq. (2.1) and separating out the real and imaginary parts of the equation, we arrive at the following coupled equations in ρ and χ,

$$k\rho + u\chi'\rho - a_1\chi'^2\rho + a_1\rho'' - a_4\chi'\rho^3 + a_2\rho^3 + a_3\rho^5 = 0, \tag{2.9}$$

and

$$-u\rho' + a_1\chi''\rho + 2a_1\chi'\rho' + (3a_4 + 2a_5)\rho^2\rho' = 0. \tag{2.10}$$

To solve these coupled equations, we have chosen the ansatz

$$\chi'(\xi) = \alpha\rho^2 + \beta. \tag{2.11}$$

Hence, chirping is given as $\delta\omega(t,z) = -(\alpha\rho^2+\beta)$, where α and β denote the nonlinear and constant chirp parameters, respectively. Using this ansatz in Eq. (2.10), we get the following relations

$$\alpha = -\frac{(3a_4 + 2a_5)}{4a_1} \quad \text{and} \quad \beta = \frac{u}{2a_1}. \tag{2.12}$$

Hence, the value of chirp parameters depend on the various coefficients of the evolution equation such as diffraction, self-steepening and self frequency shift. It means the amplitude of chirping can be controlled by varying these coefficients. Now using Eq. (2.11) and Eq. (2.12) in Eq. (2.9), we obtain

$$\rho'' + b_1\rho^5 + b_2\rho^3 + b_3\rho = 0, \tag{2.13}$$

where $b_1 = \frac{1}{16a_1^2}\left[16a_1a_3 - (2a_5 + 3a_4)(2a_5 - a_4)\right],\ b_2 = \frac{1}{2a_1^2}\left(2a_1a_2 - ua_4\right)$ and $b_3 = \frac{1}{4a_1^2}\left(4ka_1 + u^2\right)$.

This cubic-quintic nonlinear equation is known to admit a variety of solutions, like periodic, kink and solitary wave-type solutions. In general, all travelling wave solutions of Eq. (2.13) can be expressed in a generic form by means of the Weierstrass' $\wp$-function [37]. In this chapter, we have reported various localized solutions for different parameter conditions. It is interesting to note that, if $b_1 = 0$ i.e. the quintic term is related to self-steepening and self-frequency shift terms, then Eq. (2.13) reduces to cubic nonlinear equation which admits dark and bright solitons. For the case when $b_2 = 0$, it can be solved for localized solutions by using fractional transformation. For $b_3 = 0$, we show that the equation has Lorentzian type solution. In the most general case, when all the coefficients have non zero values, Eq. (2.13) can be mapped to ϕ^6 field equation to obtain double kink-type [38] as well as bright and dark soliton solutions [39].

In the following section we delineate the parameter domains in which soliton-like solutions exist for this model. For example when $b_1 = 0$, two interesting cases emerge which yield exact soliton solutions.

Case I. $b_1 = 0$

(a) For $b_2 < 0$ and $b_3 > 0$, which implies $u > \frac{2a_1a_2}{a_4}$ and $k > \frac{-u^2}{4a_1}$, one obtains a dark soliton solution of Eq. (2.13) of the form

$$\rho(\xi) = \sqrt{-\frac{b_3}{b_2}} \tanh\left(\sqrt{\frac{b_3}{2}}\xi\right). \tag{2.14}$$

The corresponding chirping is given by

$$\delta\omega(t,z) = \frac{\alpha b_3}{b_2} \tanh^2\left(\sqrt{\frac{b_3}{2}}\xi\right) - \beta. \tag{2.15}$$

(b) For $b_2 > 0$ and $b_3 < 0$, which implies $u < \frac{2a_1a_2}{a_4}$ and $k < \frac{-u^2}{4a_1}$, one can find a bright soliton solution of the form

$$\rho(\xi) = \sqrt{-\frac{2b_3}{b_2}} \operatorname{sech}\left(\sqrt{-b_3}\xi\right). \tag{2.16}$$

The chirping will be

$$\delta\omega(t,z) = \frac{2\alpha b_3}{b_2} \operatorname{sech}^2\left(\sqrt{-b_3}\xi\right) - \beta. \tag{2.17}$$

Hence, when $b_1 = 0$ i.e. $16a_1a_3 = (2a_5 + 3a_4)(2a_5 - a_4)$, the amplitude profile will be the same as for the NLSE, whereas the chirping will still show nonlinear behavior. However, unlike in the NLSE, both dark and bright solitons exist in the normal as well as anomalous dispersion regimes. But, both soliton solutions have mutually exclusive velocity space. The amplitude profile of typical dark and bright soliton is shown in Fig. 2.3, using the same values for model parameters as in Ref. [10], i.e. for $a_1 = 1.6001$, $a_2 = -2.6885$, $a_4 = 0.30814$ and $a_5 = 0.76604$. As $b_1 = 0$, quintic

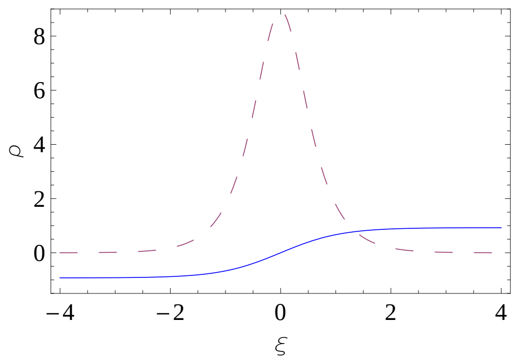

Figure 2.3: Amplitude profile for (a) dark soliton (solid line) for $u = 4.1184$ and $k = 0$; (b) bright soliton (dashed line) for $u = -30.1280$ and $k = -150.2856$.

term coefficient comes out to be $a_3 = 0.1174$. The corresponding chirping for dark and bright solitons is shown in Figs. 2.4 and 2.5, respectively (for $z = 0$). It is clear from the figure that chirping for dark soliton has a minimum at the center of the pulse whereas for bright soliton it has a maximum. But, for both cases it saturates

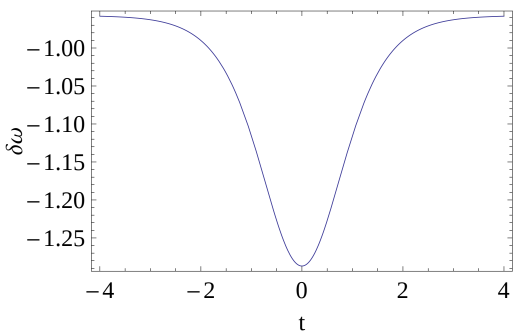

Figure 2.4: Chirping profile for dark soliton plotted in Fig. 2.3.

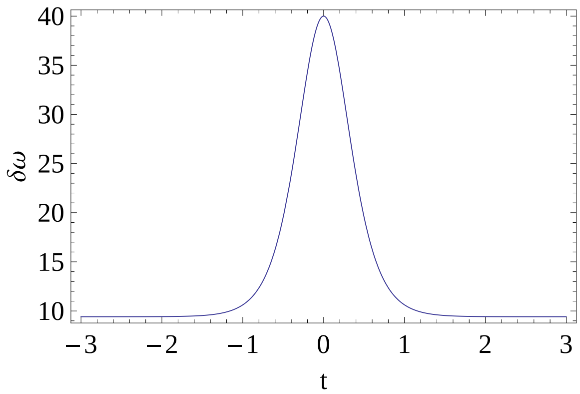

Figure 2.5: Chirping profile for bright soliton plotted in Fig. 2.3.

at same finite value as $t \to \pm\infty$.

2.5 Chirped fractional transform solitons

Case II. $b_2 = 0$

For the parametric condition $b_2 = 0$, we obtain an interesting chirped fractional transform solution. To obtain this, substitute $\rho^2 = y$ in Eq. (2.13), which then reduces to

$$y'' + \frac{8}{3}b_1 y^3 + 4b_3 y + c_0 = 0. \tag{2.18}$$

This equation can be solved for travelling wave solutions by using a fractional transformation [40]

$$y(\xi) = \frac{A + Bf^2(\xi)}{1 + Df^2(\xi)}, \tag{2.19}$$

which maps the solutions of Eq. (2.18) to the elliptic equation: $f'' \pm af \pm bf^3 = 0$, where a and b are real.

Our main aim is to study the localized solutions. We consider the case where $f = \mathrm{cn}(\xi, m)$ with modulus parameter $m = 1$, which reduces $\mathrm{cn}(\xi)$ to $\mathrm{sech}(\xi)$. We can see that Eq. (2.19) connects $y(\xi)$ to the elliptic equation, provided $AD \neq B$, and the following conditions should be satisfied for the localized solution.

$$12b_3A + 8b_1A^3 + 3c_0 = 0, \quad (2.20)$$

$$8b_3AD + 4b_3B + 4(B - AD) + 8b_1A^2B + 3c_0D = 0, \quad (2.21)$$

$$4b_3AD^2 + 8b_3BD + 4(AD - B)D + 6(AD - B) + 8b_1AB^2 + 3c_0D^2 = 0, \quad (2.22)$$

$$12b_3BD^2 + 6(B - AD)D + 8b_1B^3 + 3c_0D^3 = 0. \quad (2.23)$$

From Eq. (2.21), we find that $D = \Gamma B$, where $\Gamma = \frac{4+4b_3+8b_1A^2}{4A-8b_3A-3c_0}$. Using this in Eq. (2.22), we determine B as $B = \frac{6(1-A\Gamma)}{8b_1A+4b_3\Gamma^2A+8b_3\Gamma+4\Gamma(A\Gamma-1)+3c_0\Gamma^2}$. By substituting these expressions in Eq. (2.20) and Eq. (2.23), we can determine A and c_0 for any given values of b_1 and b_3.

So, the localized solutions are of the form

$$y(\xi) = \frac{A + B\ \mathrm{sech}^2(\xi)}{1 + D\ \mathrm{sech}^2(\xi)}, \quad (2.24)$$

and $\rho(\xi)$ can be written as

$$\rho(\xi) = \sqrt{\frac{A + B\ \mathrm{sech}^2(\xi)}{1 + D\ \mathrm{sech}^2(\xi)}}. \quad (2.25)$$

The chirping takes the form

$$\delta\omega(t, z) = -\left[\alpha\left(\frac{A + B\ \mathrm{sech}^2(\xi)}{1 + D\ \mathrm{sech}^2(\xi)}\right) + \beta\right]. \quad (2.26)$$

The typical profiles for amplitude and chirping (for $z = 0$) are shown in Fig. 2.6 and Fig. 2.7, respectively, for $a_1 = 1.6001$, $a_2 = -2.6885$, $a_3 = 0.0260$, $a_4 = 0.30814$ and $a_5 = 0.76604$, and $k = 0$. To make $b_2 = 0$, we have chosen u as $u = -27.9215$.

Case III. $b_3 = 0$

We obtain yet another interesting algebraic solution for the parametric condition $b_3 = 0$. In particular, for $b_2 < 0$ and $b_1 > 0$, the solution of Eq. (2.13) is of the following form:

$$\rho(\xi) = \frac{1}{\sqrt{M + N\xi^2}}, \quad (2.27)$$

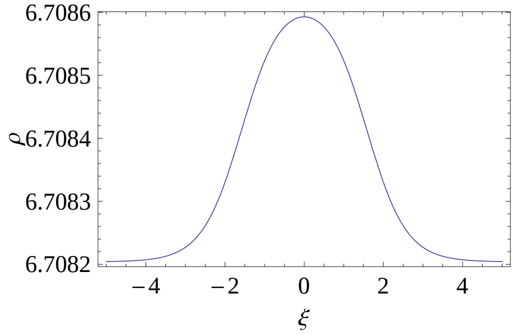

Figure 2.6: Typical amplitude profile for soliton solution given by Eq. (2.25), for values mentioned in the text.

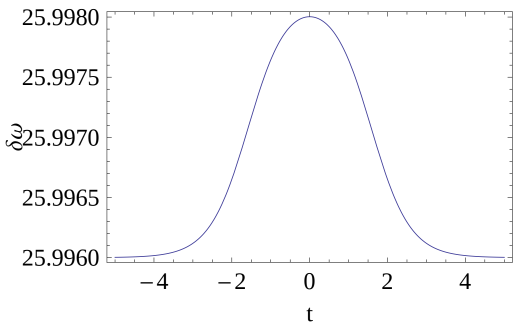

Figure 2.7: Chirping profile for soliton solution plotted in Fig. 2.6.

where $M = \frac{-2b_1}{3b_2}, N = \frac{-b_2}{2}$, and the chirping is given by

$$\delta\omega(t,z) = -\left(\frac{\alpha}{M + N\xi^2} + \beta\right). \tag{2.28}$$

For this case, the typical profiles for amplitude and chirping (for $z = 0$) are shown in Figs. 2.8 and 2.9, respectively, for $a_1 = 1.6001,\ a_2 = -2.6885,\ a_3 = 0.2174,\ a_4 = 0.30814,\ a_5 = 0.76604$, and $u = 4.1185$. For $b_3 = 0$, k can be chosen as $k = -121.8064$.

2.6 Chirped double kink and bright/dark solitons

Case IV. $b_1 \neq 0$, $b_2 \neq 0$ and $b_3 \neq 0$

We demonstrate the existence of double-kink solitons and bright/dark solitons when

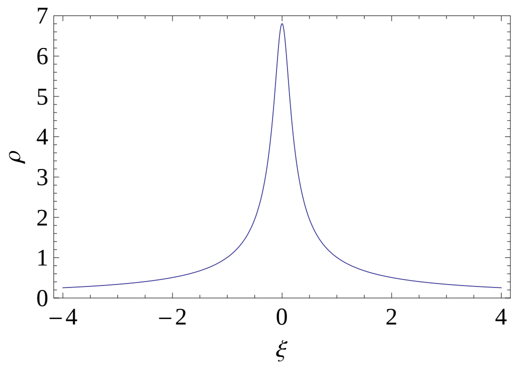

Figure 2.8: Typical amplitude profile for soliton solution given by Eq. (2.27), for values mentioned in the text.

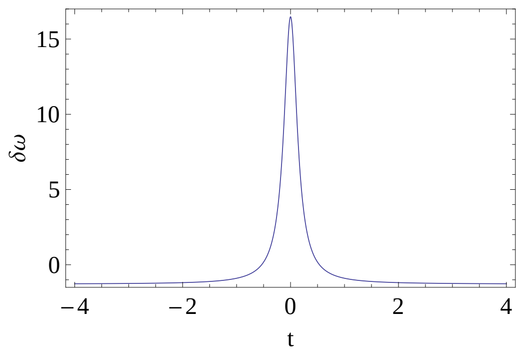

Figure 2.9: Chirping profile for soliton solution plotted in Fig. 2.8.

all the parameters in Eq. (2.13) are non zero.

(a) For the general case, Eq. (2.13) can be solved for double kink-type (usually called two kinks) soliton solutions of the form [38]

$$\rho(\xi) = \frac{p\sinh(q\xi)}{\sqrt{\epsilon + \sinh^2(q\xi)}}, \tag{2.29}$$

where $b_1 = -\frac{3q}{p}\left(\frac{\epsilon-1}{\epsilon}\right)$, $b_2 = 2pq\left(\frac{2\epsilon-3}{\epsilon}\right)$ and $b_3 = -p^3q\left(\frac{\epsilon-3}{\epsilon}\right)$. For this case, the chirping can be written as

$$\delta\omega(t,z) = -\left(\frac{\alpha p^2\sinh^2(q\xi)}{\epsilon + \sinh^2(q\xi)} + \beta\right). \tag{2.30}$$

The amplitude profile of soliton solution, for different values of ϵ is shown in Fig. 2.10, for $a_1 = 1.6001,\ a_2 = -2.6885,\ a_3 = 0.0260,\ a_4 = 0.30814,\ a_5 = 0.76604$ and

$u = -30.1280$. The interesting 'double kink' feature of the solution given by Eq. (2.29) exists only for sufficiently large values of ϵ. One can also point out that as value of ϵ changes, it only effects the width of the wave but amplitude of the wave remains the same. Chirping for solution is shown in Fig. 2.11 (for $z = 0$), which has a minimum at the center of the pulse and saturates at same finite value as $t \to \pm\infty$.

(b) It will be interesting to note that for $b_3 < 0$ and $b_2 > 0$ Eq. (2.13) has both

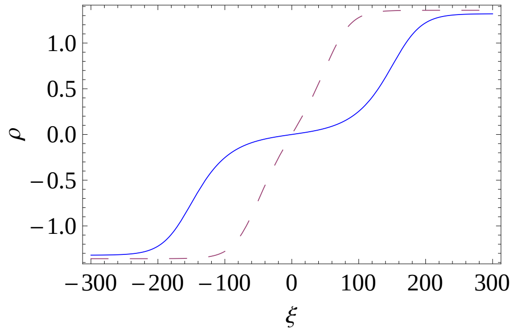

Figure 2.10: Typical amplitude profile of soliton solution in Eq. (2.29) for different values of ϵ as: (i) $\epsilon = 1000$ (solid line) for $p = 1.3204$, $q = 0.0252$ and $k = -141.911$; (ii) $\epsilon = 10$ (dashed line) for $p = 1.3584$, $q = 0.0287$ and $k = -141.899$.

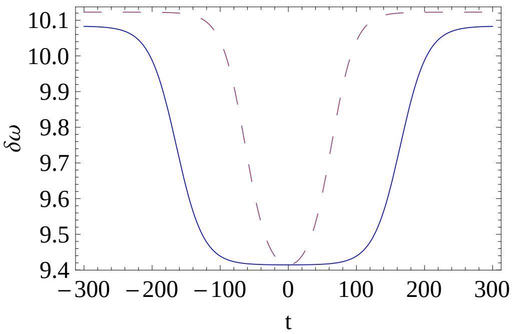

Figure 2.11: Chirping profile for soliton solutions plotted in Fig. 2.10: (i) for $\epsilon = 1000$ (solid line); (ii)for $\epsilon = 10$ (dashed line).

bright and dark solitons depending on the value of b_1 [13]. The explicit solutions and the corresponding chirping is given below:

(i) If $b_1 < |\frac{3b_2^2}{16b_3}|$, then Eq. (2.13) has bright soliton type solution which is given as

$$\rho(\xi) = \frac{p}{\sqrt{1 + r\cosh q\xi}}, \tag{2.31}$$

where $p^2 = -\frac{4b_3}{b_2}$, $q^2 = -4b_3$ and $r^2 = 1 - \frac{16b_1b_3}{3b_2^2}$. The corresponding chirping is given by

$$\delta\omega(t, z) = -\left(\frac{\alpha p^2}{1 + r\cosh q\xi} + \beta\right). \tag{2.32}$$

(ii) If $b_1 = \frac{3b_2^2}{16b_3}$, then solution of Eq. (2.13) will be of dark soliton type given by

$$\rho(\xi) = \pm p\sqrt{1 \pm \tanh q\xi}, \tag{2.33}$$

where $p^2 = -\frac{2b_3}{b_2}$ and $q^2 = -b_3$. For this case, the chirping is given as

$$\delta\omega(t, z) = -(\alpha p^2(1 \pm \tanh q\xi) + \beta). \tag{2.34}$$

In Fig. 2.12, the amplitude profile of typical bright and dark soliton is shown, for $a_1 = 1.6001$, $a_2 = -2.6885$, $a_4 = 0.30814$ and $a_5 = 0.76604$. It is interesting to note that Eq. (2.13) has bright and dark soliton depending on the value of quintic term coefficient i.e. a_3, as shown in figure. It can be seen that chirping (for $z = 0$) is also different in both cases, for bright soliton chirping has a maximum at the center of the pulse which saturates at same finite value (see Fig. 2.13), whereas for dark soliton it saturates to different finite values as $t \to \pm\infty$.

Hence, Eq. (2.13) has bright/dark soliton and double kink-type soliton solutions for same model parameters but for different velocity selection and other parameters of wave.

2.7 Conclusion

In conclusion, we would like to point out that the present work is a natural but significant generalization of Ref. [10], by considering the effect of competing cubic-quintic nonlinearity on the ensuing optical pulses in the higher order NLSE. We have demonstrated that the competing cubic-quintic nonlinearity induces propagating soliton-like (dark/bright solitons) and double kink solitons in the nonlinear

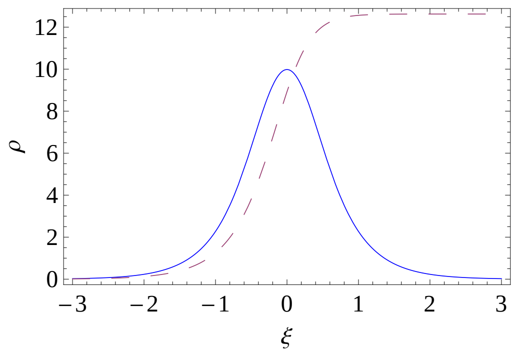

Figure 2.12: Amplitude profile of soliton solutions in Eqs. (2.31) and (2.33) for $u = -30.1280$ and $k = -150.2856$, as: (i) bright soliton (solid line) for $a_3 = 0.1168$; (ii) dark soliton (dashed line) for $a_3 = 0.1164$.

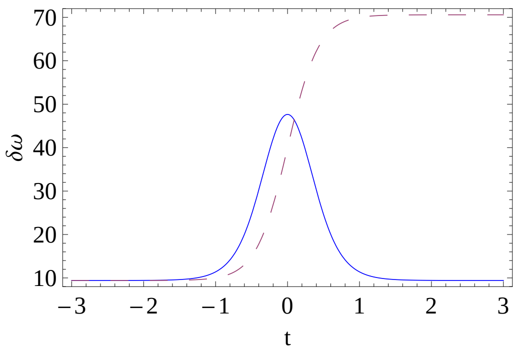

Figure 2.13: Plot for chirping for solitons plotted in Fig. 2.12: (i) for bright soliton (solid line); (ii) for dark soliton (dashed line).

Schrödinger equation with self-steepening and self-frequency shift. Parameter domains are delineated in which these optical pulses exist. Also, fractional transform solitons are explored for this model. It is observed that the nonlinear chirp associated with each of these optical pulses is directly proportional to the intensity of the wave and saturates at some finite value as the retarded time approaches its asymptotic value. We have further shown that the amplitude of the chirping can be controlled by varying self-steepening term and self-frequency shift. These optical pulses have nontrivial phase chirping which varies as a function of intensity and are different from that in Ref. [41] where the solution had a trivial phase. As the chirped pulses find significant applications in pulse compression or amplification, we hope that the newly found optical pulses may find use in the design of fiber optic compressors, optical pulse compressors, and solitary wave based communication links. Most of the work presented here has appeared in Ref. [42].

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Chapter 3

Solutions of driven nonlinear evolution equations using factorization method

3.1 Introduction

Factorization method offers a simple way to have analytical solutions of complicated nonlinear evolution equations. It is a well established technique to find a particular solution of a second order nonlinear evolution equation by reducing it into two first order equations. Exact particular solutions of several NLEEs like convective Fisher equation, generalized Burgers-Huxley equation [1], reaction-diffusion system of equation with nonlinear reaction terms [2], damped wave equations with cubic nonlinearities, modified Emden equation, KdV-Burger equation [3] and many others [4] have been obtained using this technique. This method has been further employed for solving third order differential equations [5].

This technique was developed by Schrödinger [6] in 1940 to obtain eigen functions and eigen spectra of Schrödinger equation. After that, in 1951, Infeld and Hull [7] modified this technique to handle perturbation problem, which became more widely known. In 1984, Mielnik [8] introduced factorization of quantum harmonic oscillator equation based on general Riccati solutions. As a result, he obtained a family of strictly isospectral potentials to harmonic oscillator spectrum. Construction of strictly isospectral potentials has many applications in physics, especially in context of supersymmetry [9]. The factorization technique, since then, has been extended to any solvable quantum mechanical potential. The advent of supersymmetric quantum mechanics helped in enlarging the class of exactly solvable potentials through the properties of shape invariance, quasi-exactly solvability etc. [10]. In this Chapter, we extend this factorization method to solve driven NLEEs.

The chapter is organized as follows: In Section 3.2, we discuss the formalism of factorization technique for driven NLEEs. We illustrate this by taking two specific examples in sections 3.3 − 3.6. In the last section, we summarize the results obtained in this chapter.

3.2 Factorization method

As discussed in Section 1.4, Rosu and his collaborators [3, 11] showed that if a second order non linear differential equation of the form

$$\ddot{u} + g(u)\dot{u} + F(u) = 0, \tag{3.1}$$

can be factorized as

$$[D - \phi_2(u)][D - \phi_1(u)]u = 0, \tag{3.2}$$

it becomes very easy to obtain its particular solution by solving the first order differential equation

$$[D - \phi_1(u)]u = 0. \tag{3.3}$$

We use this factorization method to solve driven NLEEs as given below.

3.2.1 Driven NLEEs with ϕ_2 as constant

Factorization technique can be extended to solve driven NLEEs whose non driven counterparts have ϕ_2 as constant.

Consider a NLEE driven by a constant force $'l'$ of the form

$$\ddot{u} + g(u)\dot{u} + F(u) = l, \tag{3.4}$$

which can be factorized as

$$[(D - \phi_2(u))][(D - \phi_1(u))]u = l. \tag{3.5}$$

If u_1 is a solution of $(D - \phi_1)u = k$, where k is another constant, then Eq. (3.5) takes the form

$$[D - \phi_2(u)]k = l, \tag{3.6}$$

and it is straightforward to see that u_1 is also a solution of Eq. (3.5) provided $k = -l/\phi_2$.

This formalism is quiet significant, as these days, work on driven NLEEs is attracting very much attention[12–14] due to a wide range of applications of the driven systems. A system is usually driven externally to balance the dissipation effect. Such systems have been explored quiet significantly in the field of nonlinear science. The driven NLEEs are used to describe the propagation of pulses in asymmetric twin-core optical fibers [15–18], charge density waves with external electric field [19], Josephson junctions [20], plasmas driven by radio frequency fields [21] etc.

3.2.2 Riccati generalization

Apart from this, Rosu-Reyes [22] found that if the first factorizing function $\phi_1(u)$ is a linear function of the dependent variable u i.e. $\phi_1(u) = c_1 u + c_2$, where c_1, c_2 are constants, then Eq. (3.3) turns out to be Riccati equation i.e.

$$\dot{u} - c_1 u^2 - c_2 u = 0. \tag{3.7}$$

If a particular solution of this equation is known, then the general solution can be written as

$$u_{\lambda,c_1} = u_1 + \frac{e^{I_1}}{\lambda - c_1 I_2}, \tag{3.8}$$

where

$$I_1(\tau) = \int_{\tau_0}^{\tau} (2c_1 u_1(\tau') + c_2) d\tau' \qquad \text{and}$$

$$I_2(\tau) = \int_{\tau_0}^{\tau} e^{I_1}(\tau') d\tau'. \tag{3.9}$$

Here λ is chosen in such a way so as to avoid singularities. It is also called 'growth parameter' in a sense that it take solution from u_1 to u_{λ,c_1}.

This technique is particularly important as in most of the nonlinear systems, we can not obtain a general solution because the linear principle of superposition is not valid for such systems. However, the Riccati parameter helps to obtain a general solution once the particular solution is known.

Here, we illustrate both of these cases by taking suitable examples.

3.3 Convective Fisher equation

Convective Fisher equation belongs to the class of convective diffusion reaction equations which describe the mathematical model for the variation of concentration of substance (pollutants, chemicals etc.) under the three processes : reaction, diffusion and convection. It is an elliptical partial differential equation which is given as

$$2\,\nu\,u_t \;-\; \mu\,u\,u_x \;=\; D\,u_{xx} \;+\; 2\,u(1-u). \tag{3.10}$$

The first term on R.H.S represents diffusion due to concentration gradient and second term represent reaction term which implies conversion of one substance to other substance, where $-u^2$ is the self limiting term depending on availability of resources. Second term on the L.H.S represent nonlinear convection which describes movement of substance with in the fluid. Here in Eq. (3.10) positive parameter μ serves to tune the relative strength of convection. It models the system in which mechanical transport competes with the diffusion and dissipation. For $\mu = 0$ this equation reduces to Fisher equation that models the systems like advances of advantageous genes [23], Brownian motion [24] and neurophysiology system [25] in the limited resources. Shönborn et al. solved Eq. (3.10) for varying μ with Kawasaki Yalabik Gunton singular perturbation method (KYGSPM) [26]. For $\mu < 1$ the results goes back to that of Fisher equation and when $\mu = O(1)$ or greater, the result breaks down due to dominating mechanical transport over the diffusion.

In the travelling frame of reference this equation reduces to an ordinary differential equation (ODE) given as

$$\ddot{u} + 2(\nu - \mu u)\dot{u} + 2u(1-u) = 0. \tag{3.11}$$

Rosu and Cornejo solved Eq. (3.11) by factorization method and obtained an explicit particular solution [11]. In the following section, we consider the corresponding driven part of convective Fisher equation and obtain topological stable particular solution as well as general solution using the factorization method discussed above.

3.4 Driven convective Fisher equation

Factorization functions for Eq. (3.11) are $\phi_1(u) = \sqrt{2}a_1(1-u)$ and $\phi_2(u) = \sqrt{2}a_1^{-1}$, where $a_1 = \frac{\mu}{\sqrt{2}}$, is a constant.

As ϕ_2 is a constant, we can solve the corresponding driven part as discussed above. If the system is driven by a constant force l, the convective Fisher equation takes the form

$$\ddot{u} + 2(\nu - \mu u)\dot{u} + 2u(1-u) = l. \tag{3.12}$$

For the same conditions on μ, Eq. (3.12) can be written as

$$[D - \sqrt{2}a_1^{-1}][D - \sqrt{2}a_1(1-u)]u = l. \tag{3.13}$$

Consider the first order differential equation

$$[D - \sqrt{2}a_1(1-u)]u = k, \tag{3.14}$$

where k is a constant. Solution of Eq. (3.14) will be solution of Eq. (3.12) for $k = -\frac{la_1}{\sqrt{2}}$.

Substituting the value of k in Eq. (3.14) and on integrating, gives one particular solution as

$$u_1 = a + b \tanh \mu b(t - t_0), \tag{3.15}$$

where $b = \frac{1}{2}\sqrt{1-2l}$; $k = \mu b^2 - \frac{\mu}{4}$; $a = \frac{1}{2}$ and $l = -\frac{2k}{\mu}$.

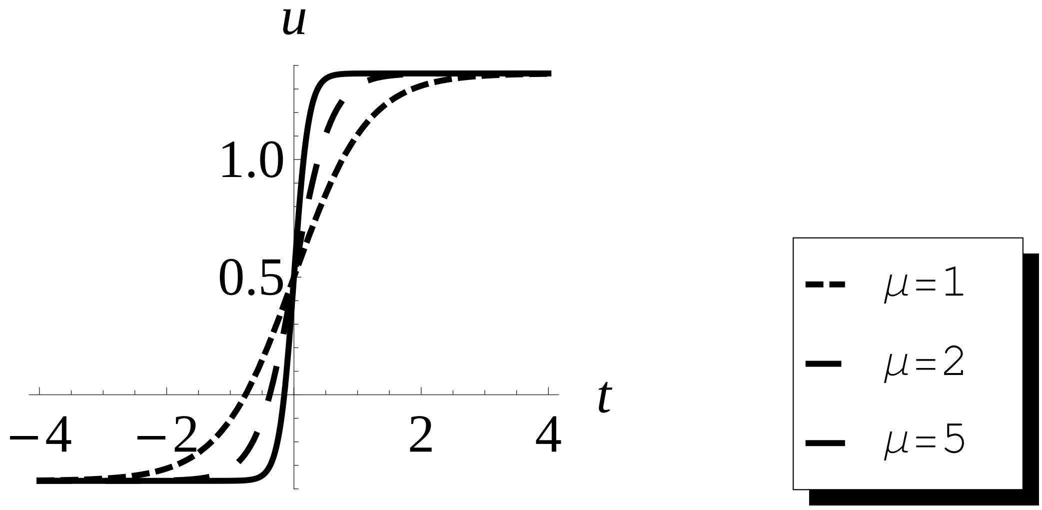

Figure 3.1: Driven Convective Fisher Equation: Graph between u and t plotted for $a = 1/2,\ \mu = 1, 2, 5,\ b = -1$.

The solution (3.15) is plotted in Figure 3.1 for different values of μ which is a parameter controlling the relative strength of convection. For the non driven case, i.e. $l = 0$, the result goes back to as given in Ref.[11].

3.4.1 Riccati generalization of solution

Since ϕ_1 is a linear function of u, so knowing the particular solution u_1, we can write down the Riccati parameterized general solution using Eq. (3.8). Substituting values of c_1, c_2 and u in (3.9) we can express I_1 and I_2 in simpler form as

$$I_1(t) = -2 \log \left[\frac{\cosh b\mu\ t}{\cosh b\mu\ t_0}\right],$$

and

$$I_2(t) = \frac{\cosh^2\, b\mu\, t_0}{b\mu}(\tanh\, b\mu\, t - \tanh\, b\mu\, t_0).$$

So, the general solution of Eq. (3.12) becomes

$$u_\lambda = a + b\ \tanh b\mu(t - t_0) + \frac{\mathrm{sech}^2\, b\mu\, t \cosh^2\, b\mu\, t_0}{\lambda + \frac{\cosh^2\, b\mu\, t_0}{b}(\tanh\ b\mu\, t - \tanh\ b\mu\, t_0)}.$$

Here λ is a Riccati parameter, it can be assigned all values except those for which denominator of solution becomes zero, so as to have its well behaved nature. The graph between u_λ and t for $a = 1/2,\ \mu = 1, 2, 5,\ b = -1$ and different values of λ is as shown in Figure 3.2.

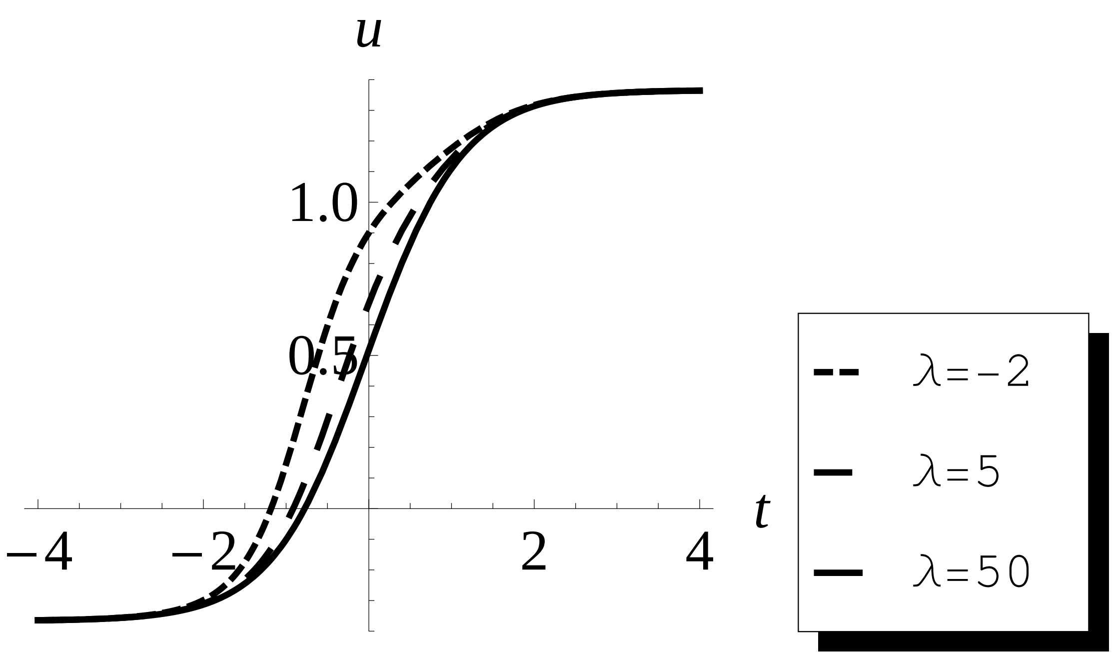

Figure 3.2: Driven Convective Fisher Equation: Graph between u_λ and t plotted for different values of λ.

3.5 Duffing-Van der Pol oscillator

Duffing-Van der Pol oscillator (DVP) equation is one of the well studied equation of nonlinear dynamics. Its autonomous version is

$$\ddot{u} + (a + bu^2)\dot{u} - \gamma u + \beta u^3 = 0, \tag{3.16}$$

where a, b, γ, β are arbitrary parameters. Eq. (3.16) is invariant under $u \to -u$. and also passes weaker form of Painlieve test. It models a typical synchronized chaotic dynamical system. A distinctive circuit realization of DVP is given by Matsumoto-Chua-Dobayashi circuit. Because of its wide range of applications it has received a lot of attention recently by many authors. In the absence of cubic nonlinearity, it reduces to the Van der Pol oscillator which was initially designed by Van der Pol to explain triode oscillator [27]. This equation is also famous, as for $a > 0, b < 0, \gamma < 0$, the existence, uniqueness and stability of special closed trajectory called a limit cycle by Poincare can be proved. For b =0, it becomes the unforced Duffing oscillator model which was originally given by Duffing to describe hardening spring effect observed in many mechanical systems. This equation has been used to study three different type of potential wells: single well potential, double well potential and double hump potential depending upon the sign of various parameters [28]. For the special case, $\gamma = 0$, Eq. (3.16) can be expressed in the form

$$\ddot{u} + g(u)\dot{u} + F(u) = 0. \tag{3.17}$$

Here $g(u) = a + bu^2$ and $F(u) = \beta u^3$. This equation is solved in [11] by the factorization method with the factorization functions $\phi_1(u) = a_1\sqrt{\beta}u^2$ and $\phi_2 = \sqrt{\beta}a_1^{-1}$. For the consistency conditions to be satisfied we must have $ab = 3\beta$ and $a_1 = \sqrt{\frac{b}{3a}}$. Since the factorizing function ϕ_2 is constant, we can solve the corresponding driven Duffing-Van der Pol equation as given below.

3.6 Driven Duffing-Van der Pol equation

With the same factorization functions as given above, the Duffing-Van der Pol oscillator driven by constant force $'l'$

$$\ddot{u} + (a + bu^2)\dot{u} + \beta u^3 = l, \tag{3.18}$$

can be rewritten as

$$[D - \sqrt{\beta}a_1^{-1}][D - \sqrt{\beta}a_1 u^2]u = l. \tag{3.19}$$

The compatible first-order differential equation is

$$[D - \sqrt{\beta}a_1 u^2]u = k, \tag{3.20}$$

where, $k = -\frac{a_1 l}{\sqrt{\beta}}$, is a constant. Substituting the value of k in Eq. (3.20) and solving it gives us the following implicit solution.

$$a_1\sqrt{\beta}(t - t_0) = -\frac{\alpha}{3l}\left(\frac{1}{2}\ln\frac{(u+\alpha)^2}{u^2 - \alpha u + \alpha^2} + \sqrt{3}\tan^{-1}\frac{u\sqrt{3}}{2\alpha - u}\right),$$

where $\alpha = -\sqrt{\frac{l}{\beta}}$.

3.7 Conclusion

In this chapter, we have seen how the factorization method can be extended further to solve the driven nonlinear evolution equations if their corresponding non driven equations can be factorized in such a manner so that ϕ_2 is a constant. Further, we have also seen the application of this method to solve the driven Convective Fisher equation as well as forced Duffing-Van der Pol oscillator where we are able to obtain topological stable particular as well as generalized solutions in the former case and implicit solution in the latter.

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Chapter 4

Nonlinear dynamics of DNA-Riccati generalized solitary wave solutions

4.1 Introduction

Deoxyribonucleic acid (DNA) molecule encodes the genetic information that organisms need to live and reproduce themselves. It not only attracts biologists but is also an attractive system for physicists because its properties can be probed very accurately by experiments that combine the methods of physics with biological tools [1]. The structure of DNA has been extensively studied during the last decades. It consists of a pair of molecules, organized as strands and joined by hydrogen as well as covalent bonds. The diameter of the double stranded DNA is 2 nm indeed, but the length could be much longer, usually. In 1953, Watson and Crick [2] firstly discovered the double helix structure of DNA, but still it is really difficult to relate all its characteristics to a specific mathematical model due to its complex structure and presence of various motions (for example, the longitudinal, transverse and torsional motions) [3]. However different motions of DNA are present on quiet different time scales, which helps to model few of these which have characteristics close to the corresponding parameters of the processes being considered and the effect of the others being considered in various parameters.

Theoretical studies of the nonlinear properties of DNA have been stimulated by the pioneer works of Davydov [4]. His idea was first of all applied to DNA by Englander and coauthors in 1980 [5], who studied the base pair opening in DNA by taking into account only the rotational motion of nitrogen bases. Their model describes the torsion dynamics of only one DNA strand which is under the action of a constant field formed by the second strand. Yomosa [6] has developed this idea further and has proposed a dynamics plane-base rotator model that was later improved by Takeno and Homma [7], in which attention was paid to the degree of freedom characterizing base rotations in the plane perpendicular to the helical axis around the backbone structure and it was shown that nonlinear molecular excitations were governed by kink-antikink solitons. Peyrard and Bishop [8] studied the process of denaturation in which only the transverse motions of bases along the hydrogen bond was taken into account. Further, Muto and coauthors [9] suggested that two types of internal motions made the main contribution to the DNA denaturation process: transverse motions along the hydrogen bond and longitudinal motions along the backbone direction. In recent years, further improvements were made in the model [10–14] to study dynamics of DNA for different motions and solitary wave type solutions were obtained. These localized nonlinear excitations explain the conformation transition [15, 16], the long range interaction of kink solitons in the double chain [17, 18], the regulation of transcription [17, 19], denaturation of DNA[8, 9] and charge transport in terms of polarons and bubbles [20].

In this chapter, we study the nonlinear dynamics of DNA for longitudinal and transverse modes and obtain a new class of solitary wave solutions. The coupled nonlinear partial differential equations, describing the nonlinear dynamics of DNA, can be reduced to a single ordinary differential equation (ODE) in the travelling wave frame which admits kink and anti-kink solutions [21]. It is known in the context of quantum mechanics [22, 23] and in general [24], that a given second order ODE can be solved by factorizing it into product of two first order operators and its Riccati generalized solution [25] can be found for a particular form of factorization. Using this method, (as discussed in methodology section in chapter 1) we have obtained one particular family of solitary kink-type solutions for different values of Riccati parameter.

4.2 Model of the DNA and equations

In this work, we consider the model used by Kong *et al.* [26] which is based on the lattice model of Peyrard and Bishop [8]. In this model, DNA molecule is supposed to consist of two long elastic strands (or rods) which represents two polynucleotide chains of the DNA molecule, connected to each other by an elastic membrane representing the hydrogen bonds between the pairs of bases in the two chains. This model takes the intermediate position between the lattice model of Peyrard and Bishop [8] and the well known Watson and Crick's [2] model. We assume a homogenous DNA molecule, therefore both strands have same mass density. In order to make the work self-contained, we sketch the essential steps of [26] and to simplify the calculations, we shall make two assumptions. First, we neglect the helical structure of DNA. So, instead of the double helix we shall consider two parallel strands, each having the form of a straight line. In the cases, where the effects of helicity are important, the helical structure of the DNA molecule can be taken into account [27]. Second, we consider the longitudinal and transverse motions of DNA strands, and neglect the torsional motion. In this respect our work differs from that in Ref.[26], where in only the longitudinal motion was considered. Therefore, the model includes four degrees of freedom, u_1, v_1 and u_2, v_2, for the two strands respectively. The u_1 and u_2 represent respectively the longitudinal displacements of the top strand and the bottom strand i.e. the displacement of the bases from their equilibrium positions along the direction of the phosphodiester bridge that connects the two bases of the same strand; v_1 and v_2 denote respectively the transverse displacements of the top strand and the bottom strand i.e. the displacement of the bases from their equilibrium positions along the direction of the hydrogen bonds that connect the two bases of the base pair.

The Hamiltonian of such a model can be written as

$$H = T + V_1 + V_2. \tag{4.1}$$

where T is the kinetic energy of the elastic strands, V_1 is the potential energy of the elastic strands and V_2 is the potential energy of the elastic membrane. Here

$$T = \int \frac{1}{2}\rho\sigma(\dot{u_1}^2 + \dot{u_2}^2 + \dot{v_1}^2 + \dot{v_2}^2) \;\; dx, \tag{4.2}$$

$$V_1 = \int \frac{1}{2}Y\sigma\left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial x}\right)^2\right] + \frac{1}{2}F\sigma\left[\left(\frac{\partial v_1}{\partial x}\right)^2 + \left(\frac{\partial v_2}{\partial x}\right)^2\right] \;\; dx, \tag{4.3}$$

and

$$V_2 = \int \frac{1}{2}\mu(\Delta l(x))^2 \;\; dx, \tag{4.4}$$

where ρ, σ, Y and F denote respectively the mass density, the area of transverse cross-section, the Young's modulus and the tension density of each strand; μ is the rigidity of the elastic membrane and $\Delta l(x)$ is the stretching of the elastic membrane at x due to longitudinal vibrations and is given by

$$\Delta l = \sqrt{(h + v_1 - v_2)^2 + (u_2 - u_1)^2} - l_0. \tag{4.5}$$

where h is the distance between the two strands and l_0 is the height of the membrane in the equilibrium position.

The dynamical equations associated with Hamiltonian can be written as

$$\begin{aligned}
\sigma\left[\rho\frac{\partial^2 u_1}{\partial t^2} - Y\frac{\partial^2 u_1}{\partial x^2}\right] &= \mu\tfrac{\Delta l}{l_0+\Delta l}(u_2 - u_1),\\
\sigma\left[\rho\frac{\partial^2 u_2}{\partial t^2} - Y\frac{\partial^2 u_2}{\partial x^2}\right] &= \mu\tfrac{\Delta l}{l_0+\Delta l}(u_1 - u_2),\\
\sigma\left[\rho\frac{\partial^2 v_1}{\partial t^2} - F\frac{\partial^2 v_1}{\partial x^2}\right] &= \mu\tfrac{\Delta l}{l_0+\Delta l}(v_2 - v_1 - h),\\
\sigma\left[\rho\frac{\partial^2 v_2}{\partial t^2} - F\frac{\partial^2 v_2}{\partial x^2}\right] &= \mu\tfrac{\Delta l}{l_0+\Delta l}(v_1 - v_2 + h).
\end{aligned} \tag{4.6}$$

From Eq. (4.5), we have

$$\frac{\Delta l}{l_0 + \Delta l} = \frac{\sqrt{(h + v_1 - v_2)^2 + (u_2 - u_1)^2} - l_0}{\sqrt{(h + v_1 - v_2)^2 + (u_2 - u_1)^2}}. \tag{4.7}$$

Assuming $|u_1 - u_2| << h,\ |v_1 - v_2| << h$ and neglecting higher order (> 2) terms, Eq. (4.7) can be written as

$$\frac{\Delta l}{l_0 + \Delta l} = R(u_1, u_2, v_1, v_2) = 1 - \frac{l_0}{h} + \frac{l_0}{h^2}(v_1 - v_2) - \frac{l_0}{h^3}(v_1 - v_2)^2 + \frac{l_0}{2h^3}(u_2 - u_1)^2. \quad (4.8)$$

Then Eq. (4.6) reduces to

$$\begin{aligned}
\sigma\left[\rho\frac{\partial^2 u_1}{\partial t^2} - Y\frac{\partial^2 u_1}{\partial x^2}\right] &= \mu R(u_1, u_2, v_1, v_2)(u_2 - u_1), \\
\sigma\left[\rho\frac{\partial^2 u_2}{\partial t^2} - Y\frac{\partial^2 u_2}{\partial x^2}\right] &= \mu R(u_1, u_2, v_1, v_2)(u_1 - u_2), \\
\sigma\left[\rho\frac{\partial^2 v_1}{\partial t^2} - F\frac{\partial^2 v_1}{\partial x^2}\right] &= \mu R(u_1, u_2, v_1, v_2)(v_2 - v_1 - h), \\
\sigma\left[\rho\frac{\partial^2 v_2}{\partial t^2} - F\frac{\partial^2 v_2}{\partial x^2}\right] &= \mu R(u_1, u_2, v_1, v_2)(v_1 - v_2 + h).
\end{aligned} \quad (4.9)$$

4.3 Solitary wave solutions

We can separate Eq. (4.9) into in-phase and out-of-phase motion by changing the variables as $u_+ = \frac{(u_1+u_2)}{\sqrt{2}}$ and $v_+ = \frac{(v_1+v_2)}{\sqrt{2}}$ for in-phase motion and $u_- = \frac{(u_2-u_1)}{\sqrt{2}}$ and $v_- = \frac{(v_2-v_1)}{\sqrt{2}}$ for out-of-phase motion. As only the out- of-phase motion stretches the hydrogen bond [8, 28], we will consider only out-of-phase motion for which equation of motion can be rewritten from Eq. (4.9) as

$$\begin{aligned}
\frac{\partial^2 u_-}{\partial t^2} - c_1^2\frac{\partial^2 u_-}{\partial x^2} &= \lambda_1 u_- + \gamma_1 u_- v_- + \mu_1 u_-^3 + \beta_1 u_- v_-^2, \\
\frac{\partial^2 v_-}{\partial t^2} - c_2^2\frac{\partial^2 v_-}{\partial x^2} &= \lambda_2 v_- + \gamma_2 u_-^2 + \mu_2 u_-^2 v_- + \beta_2 v_-^3 + c_0,
\end{aligned} \quad (4.10)$$

where

$$c_1 = \pm\sqrt{\frac{Y}{\rho}};\ c_2 = \pm\sqrt{\frac{F}{\rho}};\ \lambda_1 = \frac{-2\mu}{\rho\sigma h}(h - l_0);\ \lambda_2 = \frac{-2\mu}{\rho\sigma};\ \gamma_1 = 2\gamma_2 = \frac{2\sqrt{2}\mu l_0}{\rho\sigma h^2};$$
$$\mu_1 = \mu_2 = \frac{-2\mu l_0}{\rho\sigma h^3};\ \beta_1 = \beta_2 = \frac{4\mu l_0}{\rho\sigma h^3};\ c_0 = \frac{\sqrt{2}\mu(h - l_0)}{\rho\sigma}. \quad (4.11)$$

Introducing the transformation $v_- = au_- + b$ (a and b are constants), in Eq. (4.10) gives us

$$\frac{\partial^2 u_-}{\partial t^2} - c_1^2 \frac{\partial^2 u_-}{\partial x^2} = u_-^3(\mu_1 + \beta_1 a^2) + u_-^2(2\beta_1 ab + a\gamma_1) + u_-(\lambda_1 + b\gamma_1 + \beta_1 b^2). \quad (4.12)$$

$$\frac{\partial^2 u_-}{\partial t^2} - c_2^2 \frac{\partial^2 u_-}{\partial x^2} = u_-^3(\mu_2+\beta_2 a^2) + u_-^2\left(\frac{\gamma_2}{a} + \frac{\mu_2 b}{a} + 3\beta_2 ab\right) + u_-(\lambda_2+3\beta_2 b^2) + \frac{\lambda_2 b}{a} + \frac{\beta_2 b^3}{a} + \frac{c_0}{a}. \quad (4.13)$$

Comparing Eq. (4.12) and Eq. (4.13) and by using Eq. (4.11), we have $b = \frac{h}{\sqrt{2}}$ and $F = Y$. Renaming u_- as ϕ, Eq. (4.12) and Eq. (4.13) can be written as

$$\frac{\partial^2 \phi}{\partial t^2} - c_1^2 \frac{\partial^2 \phi}{\partial x^2} = A\phi^3 + B\phi^2 + C\phi + D, \quad (4.14)$$

where

$$A = \left(\frac{-2\alpha}{h^3} + \frac{4a^2\alpha}{h^3}\right); \; B = \frac{6\sqrt{2}a\alpha}{h^2}; \; C = \left(\frac{-2\alpha}{l_0} + \frac{6\alpha}{h}\right); \; D = 0; \;\; \text{with} \;\; \alpha = \frac{\mu l_0}{\rho\sigma}. \quad (4.15)$$

In travelling frame, Eq.(4.14) can be written as

$$\phi_{\xi\xi} = A\phi^3 + B\phi^2 + C\phi, \quad (4.16)$$

where $\xi = \frac{x}{c_1} - \sqrt{2}\, t$. In order to solve Eq. (4.16), let's assume

$$\phi = M\left(\frac{y'}{y}\right) + N, \quad (4.17)$$

where M and N are constants, and y satisfies the elliptic equation

$$y'' = (p + qy^2)y. \quad (4.18)$$

From Eq. (4.17) and Eq. (4.18), we get the following equation

$$\phi'' = \frac{2}{M^2}\phi^3 - \frac{6N}{M^2}\phi^2 + \left(\frac{6N^2}{M^2} - 2p\right)\phi + \left(2Np - \frac{2N^3}{M^2}\right). \quad (4.19)$$

Comparing Eq. (4.16) and Eq. (4.19), we have

$$M = \pm\sqrt{\frac{h^3}{\alpha(2a^2-1)}}; N = -\frac{\sqrt{2}ha}{(2a^2-1)}; \; p = \frac{N^2}{M^2} = \frac{2a^2\alpha}{h(2a^2-1)}; \;\; \text{with} \;\; a^2 = \frac{3l_0 - h}{2(l_0 - h)}, \quad (4.20)$$

as $p = \frac{N^2}{M^2} \Rightarrow p > 0$ and $a^2 > \frac{1}{2}$.

From Eq. (4.18), $y = R\ \text{sech}(m\xi)$, where $R^2 = -\frac{2p}{q}$ and $m^2 = p$. Then Eq. (4.16) has the following exact solitary wave solution

$$u_- \equiv \phi = M\left(\frac{y'}{y}\right) + N = -M\sqrt{p}\ \tanh(\sqrt{p}\xi) + N. \tag{4.21}$$

Substituting for M, N and p from Eq. (4.20), we have

$$u_- = \frac{-\sqrt{2}ah}{(2a^2-1)}[1 \pm \tanh(\sqrt{\frac{2a^2\mu l_0}{\rho\sigma h(2a^2-1)}}\xi)]. \tag{4.22}$$

in which, the plus or minus sign stands for anti-kinks or kinks respectively, and $a^2 > \frac{1}{2}$ and $\xi = \frac{1}{c_1}(x - \sqrt{2}c_1 t)$ such that $c_1 = \pm\sqrt{\frac{Y}{\rho}}$. It is interesting to note that the amplitude and width of the wave is arbitrary modulo with $a^2 > \frac{1}{2}$.

4.4 Riccati generalization of solution

Rosu *et. al.* [24] offer a simple way to solve a second order nonlinear differential equation by using factorization method. If a second order differential equation can be factorized to following form

$$[D - f_2(\phi)]\,[D - f_1(\phi)]\,\phi = 0, \tag{4.23}$$

then the particular solution of Eq. (4.23) can be easily found by solving the first order differential equation

$$[D - f_1(\phi)]\,\phi = 0. \tag{4.24}$$

Reyes and Rosu [25] extended this work and found an interesting result that if f_1 is a linear function of ϕ, then Eq. (4.24) turns out to be Riccati equation and knowing the particular solution ϕ_1 of Eq. (4.24) one can obtain its Riccati parameterized general solution $\phi_{\lambda,p}$, which is given as follow

$$\phi_{\lambda,p} = \phi_1 + \frac{e^{I_1}}{\lambda - pI_2}, \tag{4.25}$$

where

$$
\begin{aligned}
I_1(\xi) = &\int_{\xi_0}^{\xi} (2p\phi_1(\xi') + q)\ d\xi', \\
I_2(\xi) = &\qquad \int_{\xi_0}^{\xi} e^{I_1}(\xi')\ d\xi',
\end{aligned}
\tag{4.26}
$$

and λ is known as Riccati parameter which is to be chosen in such a way so as to avoid singularities. It is also called 'growth parameter' in a sense that it takes the solution from ϕ_1 to $\phi_{\lambda,p}$.

It can be shown that Eq. (4.16) can be factorized as Eq. (4.23), such that

$$
f_1(\phi) = p\phi + q \quad \text{and} \quad f_2(\phi) = -(2p\phi + q),
\tag{4.27}
$$

where p and q are constants which satisfy the following relations

$$
p^2 = \frac{A}{2} \quad \text{and} \quad q^2 = C,
\tag{4.28}
$$

with a constraining condition $B^2 = \frac{9}{2}AC$, which implies $a^2 = \frac{3l_0 - h}{2(l_0 - h)}$. Consider the first order differential equation

$$
\left[D - f_1(\phi)\right]\phi = 0,
\tag{4.29}
$$

whose solution will also be solution of Eq. (4.23). Integration of Eq. (4.29) gives us

$$
\phi = -\frac{q}{2p}\left[1 \pm \tanh\left(\frac{q}{2}\xi\right)\right].
\tag{4.30}
$$

Substituting the values of p and q from Eq. (4.28), we can write one particular solution for Eq. (4.16) as

$$
\phi = -\frac{B}{3A}\left[1 \pm \tanh\left(\frac{\sqrt{C}}{2}\xi\right)\right].
\tag{4.31}
$$

By substituting the values of A, B and C from Eq. (4.15), we get the same solution for Eq. (4.16) as we get before by using elliptic function method.

As f_1 comes out to be a linear function of ϕ, we can write down the Riccati parameterized general solution for Eq. (4.23). Consider the solution (4.31) with positive sign

$$
\phi_1 = -\frac{B}{3A}\left[1 + \tanh\left(\frac{\sqrt{C}}{2}\xi\right)\right],
\tag{4.32}
$$

and use Eq. (4.28), to write I_1 and I_2 from Eq. (4.26) as

$$I_1 = -2\ln\left[\frac{\cosh\left(\frac{\sqrt{C}}{2}\xi\right)}{\cosh\left(\frac{\sqrt{C}}{2}\xi_0\right)}\right], \tag{4.33}$$

$$I_2 = \frac{2}{\sqrt{C}}\cosh^2\left(\frac{\sqrt{C}}{2}\xi_0\right)\left[\tanh\left(\frac{\sqrt{C}}{2}\xi\right) - \tanh\left(\frac{\sqrt{C}}{2}\xi_0\right)\right], \tag{4.34}$$

and Riccati generalized solution for Eq. (4.16) as

$$\phi_\lambda = -\frac{B}{3A}\left[1+\tanh\left(\frac{\sqrt{C}}{2}\xi\right)\right] + \frac{\cosh^2\left(\frac{\sqrt{C}}{2}\xi_0\right)\operatorname{sech}^2\left(\frac{\sqrt{C}}{2}\xi\right)}{\lambda - \frac{3A}{B}\cosh^2\left(\frac{\sqrt{C}}{2}\xi_0\right)\left[\tanh\left(\frac{\sqrt{C}}{2}\xi\right) - \tanh\left(\frac{\sqrt{C}}{2}\xi_0\right)\right]}. \tag{4.35}$$

The graphical representation of typical generalized solution for different values of λ is shown in Figure 4.1, which confirms that Eq. (4.35) is a topological stable solution.

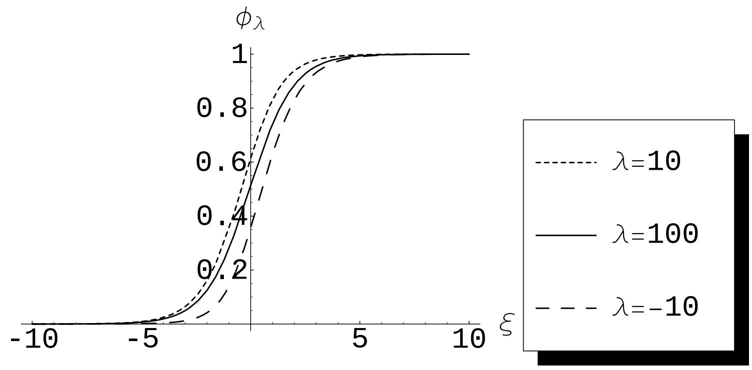

Figure 4.1: Graphical representation of ϕ_λ for different values of λ

4.5 Conclusion

In this chapter, we studied the nonlinear model for DNA by taking into account the longitudinal and transverse motions of DNA. The pair of coupled nonlinear partial differential equations describing the model, were decoupled by choosing a linear relation between longitudinal and transverse motions. We found the particular solution and then obtained Riccati generalized solution of solitary kink-type. The Riccati parameter 'λ' plays the role of growth parameter. The results presented here are a part of the work that appeared in Ref. [29].

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Chapter 5

Solitary wave solutions of nonlinear reaction-diffusion equations with variable coefficients

5.1 Introduction

Reaction-diffusion (RD) systems are mathematical models which explain how the concentration of one or more substances distributed in space changes under the influence of these two processes. Coupled systems of reaction-diffusion equations have recently attracted considerable attention, as these equations can be used to model the evolution problems in real world [1–4] like the population density, mass concentration, neuron flux, temperature etc. These equations appear in many branches of physics, chemistry, biology, ecology and engineering, because the processes of reaction and diffusion each play essential roles in the dynamics of many such systems. The study of these equations is more interesting due to the presence of nonlinear reaction term and rich variety of properties of their solutions.

Exact travelling wave solutions, specially solitary wave-type solutions, to nonlinear reaction diffusion (NLRD) equations play an important role in the qualitative description of many phenomena such as flow in porous media [5–7], heat conduction in plasma [8, 9], combustion problems [10, 11], liquid evaporation [12], population genetics [13] and image processing [14]. Hence, exact solutions to these equations play an essential role in the qualitative description of many phenomena and processes. There are many numerical and analytical techniques to find exact solutions of these equations, such as the Lie symmetry [15–17], the ansatz-based method [17–19], the Galilean-invariant method [20], the Painlevé analysis [21, 22], sign-invariant theory [23] and the algebraic method [24, 25].

RD equation describes the evolution of a system under the influence of diffusion and reaction. In the assembly of particles, for example cells, bacteria, chemicals, animals and so on, each particle moves in random way. This microscopic irregular movement results in some macroscopic regular motion of the group of particles which is called diffusion process. The gross movement is not a simple diffusion, but one has to consider the interaction between the particles and environment, which results the production of new particles. This constitutes the reaction process. Time evolution of a system under these two effects can be written as

$$u_t = Du_{xx} + f(u), \tag{5.1}$$

where $u(x, t)$ represents the concentration of the substance, D is diffusion coefficient, $f(u)$ represents the reaction term. The reaction term can be either linear or nonlinear depending upon the system. If it is a linear term, then it can be solved easily using variable separable method. But, if reaction term is nonlinear, then these nonlinear reaction diffusion (NLRD) equations can not be solved using direct methods.

NLRD equations with convection term

In a reaction diffusion system, generally, it is assumed that the field is fixed i.e. spatial transport is only through diffusion. But, in real world phenomena, the field itself usually moves. Hence, in many processes, in addition to diffusion, motion can also be due to advection or convection with some kind of back reaction, such as the spread of a favored gene, ecological competition and so on [26]. Nonlinear convection terms arise naturally, for example, in the motion of chemotactic cells [27].

From a physical point of view convection, diffusion and reaction processes are quite fundamental to describe a wide variety of problems in physical, chemical, biological sciences [28]. A general form of such NLRD equation with convection term is

$$u_t + v\ u^m u_x = D u_{xx} + f(u), \tag{5.2}$$

where v is convection coefficient and m is a real number.

NLRD equations with variable coefficients

The literature discussing the NLRD equations is massive, but these results assume that the environment is temporally and spatially homogeneous. However, this may be a rough approximation to many systems, because most of physical and biological systems are inhomogeneous due to fluctuations in environmental conditions and non-uniform media. Hence, most of real nonlinear physical equations possess variable coefficients, both in space and over time [29–31]. For example, to describe the propagation of pulses in inhomogeneous media with non uniform boundaries, we need variable-coefficient nonlinear equations due to variations of both of dispersive and nonlinear effects. The effect of spatial inhomogeneities on NLRD systems [26, 32, 33] has been discussed by many authors, but the effect of temporal inhomogeneities has not been much explored. We shall work on this problem here. There are many NLRD systems where the relevant parameters are time dependent [34, 35] because external factors make the density and/or temperature change in time. For example, in biological applications, such as population range expansion, for which reproductive (reactive) and mobility (diffusive) parameters change in time driven by climatic changes.

The dimensionless form of the NLRD equation with time-dependent coefficients of convective term and reaction term, that is studied in this chapter, is given by

$$u_t + v(t) u_x = D u_{xx} + \alpha(t) u - \beta u^n. \tag{5.3}$$

where $u = u(x,t)$, is the concentration or density variable depending on the phenomena under study; D is diffusion coefficient; v is convective term coefficient; α, β are reaction term coefficients and n is a positive integer such that $n \geq 2$. In Eq. (5.3), first term on right hand side represent diffusion due to concentration gradient and last term characterizes the nonlinearity of the system. On left hand side, first term shows the evolution of system with time and second term represents the convective flux term.

In this chapter, we have considered Eq. (5.3) for $m = 0$ and $n = 2$ and $n = 3$ i.e. for quadratic and cubic nonlinearities for two different cases, (i) only v is time dependent, (ii) both v and α are time dependent by using the auxiliary equation method [37] as discussed in Section 1.4.

5.2 Exact solutions of Eq. (5.3) for $n = 2$

Case (i) v ***is time dependent***

For this case, Eq. (5.3) reads

$$u_t + v(t)u_x = Du_{xx} + \alpha u - \beta u^2. \tag{5.4}$$

We assume that the solution of Eq. (5.4) is of the following form

$$u(\xi) = a + b\ \phi(\xi) + c\ \phi^2(\xi), \tag{5.5}$$

where a, b, c are functions of time to be determined; $\xi = kx + \eta(t)$, k is constant, and ϕ should satisfy an ordinary differential equation, viz.

$$(\phi_\xi)^2 = p\phi^2 + q\phi^3 + r\phi^4, \tag{5.6}$$

where p, q, r are constants. By substituting Eqs. (5.5) and (5.6) in Eq. (5.4), and then setting the coefficients of $\phi^i (i = 0, 1, 2, 3, 4), \phi_\xi$ and $\phi\phi_\xi$ equal to zero, we get the following set of algebraic equations for a, b, c, k and η :

$$
\begin{aligned}
&- 6crDk^2 + \beta c^2 = 0, \\
&- 2brDk^2 - 5cqDk^2 + 2bc\beta = 0, \\
&2ac\beta + b^2\beta + c_t - \frac{3}{2}bqDk^2 - 4cpDk^2 - \alpha c = 0, \\
&2ab\beta + b_t - bpDk^2 - \alpha b = 0, \\
&b\eta_t + vbk = 0, \\
&2c\eta_t + 2vck = 0, \\
&a^2\beta + a_t - \alpha a = 0. \qquad (5.7)
\end{aligned}
$$

Solving these equations consistently, we get

$$
\begin{aligned}
&a = \frac{\alpha}{\beta},\ b = \frac{3q\alpha}{p\beta},\ c = \frac{6r\alpha}{p\beta}, \\
&k = \pm\sqrt{\frac{\alpha}{pD}},\ \eta(t) = -k\int v(t)\ dt, \qquad (5.8)
\end{aligned}
$$

along with a constraining condition $q^2 = 4pr$. From Eq. (5.8) it is clear that a, b, c are constants. As $q^2 = 4pr$, the general solution for Eq. (5.6) can be obtained by quadrature method in a straightforward way as [36]

$$
\phi(\xi) = -\frac{q}{r}\left\{\frac{\operatorname{sech}^2\left(\frac{\sqrt{p}}{2}\xi\right)}{4 - \left[1 - \tanh\left(\frac{\sqrt{p}}{2}\xi\right)\right]^2}\right\}, \qquad (5.9)
$$

and finally, we get the solution for Eq. (5.4) using Eq. (5.5) as

$$
\begin{aligned}
u(x,t) = \frac{\alpha}{\beta} - \frac{12\alpha}{\beta}&\left\{\frac{\operatorname{sech}^2\left(\frac{\sqrt{p}}{2}\xi\right)}{4 - \left[1 - \tanh\left(\frac{\sqrt{p}}{2}\xi\right)\right]^2}\right\} \\
- \frac{6q\alpha}{p\beta}&\left\{\frac{\operatorname{sech}^2\left(\frac{\sqrt{p}}{2}\xi\right)}{4 - \left[1 - \tanh\left(\frac{\sqrt{p}}{2}\xi\right)\right]^2}\right\}^2, \qquad (5.10)
\end{aligned}
$$

where $\xi = k\left(x - \int v(t)\ dt\right)$. It means, non-trivial time dependance of the ξ variable is entirely expressed in terms of function of $v(t)$. Typical profile of Eq. (5.10) is shown in Figure 5.1, for $\alpha = 1,\ \beta = 1,\ D = 1,\ p = 1,\ q = 1$ and $v(t) = \sin(2t)$.

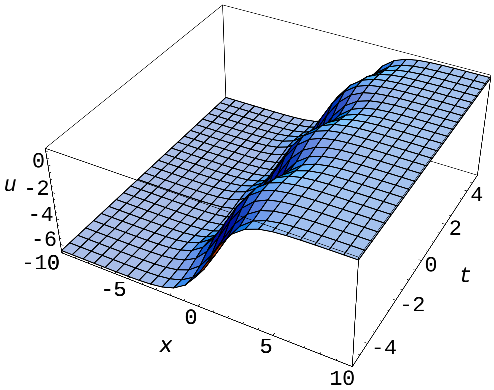

Figure 5.1: Typical form of u(x,t), Eq. (5.10) for values mentioned in the text.

It is interesting to note that for small values of x the function $u(x,t)$ has periodic structure in time and for large values of x and t, $u(x,t)$ reaches a constant value.

Case (ii) v ***and*** α ***both are time dependent***

In order to solve Eq. (5.4) for the case where both v and α are time dependent, we follow the same procedure as in last section and get the same set of algebraic equations as Eq. (5.7) with $v(t)$ and $\alpha(t)$. Solving these equations consistently, we get the following relations

$$\begin{aligned} b &= \frac{3qDk^2}{\beta}, \ c = \frac{6rDk^2}{\beta}, \\ a(t) &= \frac{pDk^2}{2\beta}[1 - \tanh(\gamma t)], \\ \alpha(t) &= -pDk^2 \tanh(\gamma t), \ \eta(t) = -k \int v(t)\, dt, \end{aligned} \tag{5.11}$$

where $\gamma = \frac{pDk^2}{2}$ and $q^2 = 4pr$. Here b, c are constants but a turns out to be a function of time. The complete solution $u(\xi)$ for Eq. (5.5) reads

$$\begin{aligned} u(\xi) =& \frac{pDk^2}{2\beta}\left(1 - \tanh(\gamma t)\right) \\ &- \frac{12pDk^2}{\beta}\left\{\frac{\operatorname{sech}^2\left(\frac{\sqrt{p}}{2}\xi\right)}{4 - \left[1 - \tanh\left(\frac{\sqrt{p}}{2}\xi\right)\right]^2}\right\} \\ &- \frac{6qDk^2}{\beta}\left\{\frac{\operatorname{sech}^2\left(\frac{\sqrt{p}}{2}\xi\right)}{4 - \left[1 - \tanh\left(\frac{\sqrt{p}}{2}\xi\right)\right]^2}\right\}^2, \end{aligned} \tag{5.12}$$

where $\xi = k\left(x - \int v(t)\ dt\right)$. Typical profile of Eq. (5.12) is shown in Figure 5.2, for $\beta = 1,\ D = 1,\ k = 1,\ p = 1,\ q = 1$ and $v(t) = e^t$. Notice that the coefficient of u in Eq. (5.4) changes sign as t changes its sign.

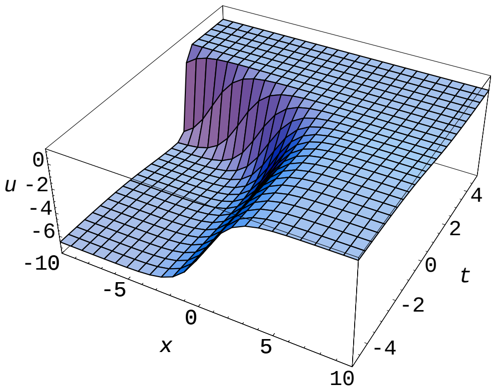

Figure 5.2: Typical form of u(x,t), Eq. (5.12) for values mentioned in the text.

5.3 Exact solutions of Eq. (5.3) for $n = 3$

Case (i) v ***is time dependent***

Eq. (5.3) reads

$$u_t + v(t)u_x = Du_{xx} + \alpha u - \beta u^3. \tag{5.13}$$

We assume that the solution of Eq. (5.13) is of the following form

$$u(\xi) = a + b\ \phi(\xi), \tag{5.14}$$

where a, b functions of time to be determined; $\xi = kx + \eta(t)$, k is constant, and ϕ should satisfy an ordinary differential equation, viz.

$$\phi_\xi = p - \phi^2, \tag{5.15}$$

where p is constant. By substituting Eq. (5.14) and Eq. (5.15) in Eq. (5.13), and then setting the coefficients of $\phi^i (i = 0, 1, 2, 3)$ equal to zero, we get the following set of algebraic equations for a, b, k and η :

$$\begin{aligned}
&-2bDk^2 + \beta b^3 = 0, \\
&-b\eta_t - bkv + 3ab^2\beta = 0, \\
&b_t + 2pbDk^2 + 3a^2b\beta - \alpha b = 0, \\
&a_t + pb\eta_t + pbkv + \beta a^3 - \alpha a = 0.
\end{aligned} \tag{5.16}$$

Solving these equations consistently, we get

$$\begin{aligned}
a^2 &= \frac{\alpha}{4\beta}\ ,\ b^2 = \frac{2Dk^2}{\beta} = \frac{\alpha}{4p\beta}, \\
k^2 &= \frac{\alpha}{8pD},\ \eta(t) = 3ab\beta t - k\int v(t)\ dt.
\end{aligned} \tag{5.17}$$

Eq. (5.17) indicates that a, b are constants. By integrating Eq. (5.15), we get the solution for $\phi(\xi)$ as

$$\phi(\xi) = \sqrt{p}\tanh(\sqrt{p}\xi), \tag{5.18}$$

and finally we get the solution for Eq. (5.13) by using Eq. (5.14) as

$$u(\xi) = \pm\sqrt{\frac{\alpha}{4\beta}}[1 + \tanh(\sqrt{p}\xi)], \tag{5.19}$$

where $\xi = kx + 3ab\beta t - k\int v(t)\ dt$. Typical profile of Eq. (5.19) is shown in Figure 5.3, for $\alpha = 1,\ \beta = 1/4,\ D = 1/2,\ p = 1$ and $v(t) = \tanh(t)$.

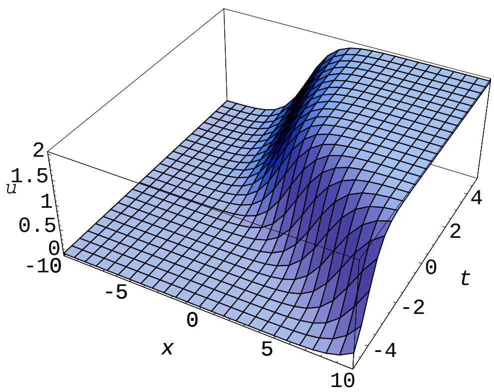

Figure 5.3: Typical form of u(x,t), Eq. (5.19) for values mentioned in the text.

Case (ii) v ***and*** α ***both are time dependent***

In order to solve Eq. (5.13) for the case where both v and α are time dependent, we follow the same procedure as in last section and get the same set of algebraic equations as Eq. (5.16) with $v(t)$ and $\alpha(t)$. Solving these equations consistently, we get the following relations

$$b^2 = \frac{2Dk^2}{\beta}\ , \alpha = pDk^2[5 \pm 3\tanh(\gamma t)],$$
$$a^2(t) = \frac{pDk^2}{\beta}[1 \pm \tanh(\gamma t)],$$
$$\eta(t) = 3ab\beta t - k \int v(t)\ dt, \tag{5.20}$$

where $\gamma = \mp 4pDk^2$. Here b is constant and a is function of time. The complete solution for $u(\xi)$, Eq. (5.14) reads

$$u(\xi) = \pm\sqrt{\frac{2pDk^2}{\beta}}\left[\sqrt{\frac{1 \pm \tanh(\gamma t)}{2}} + \tanh(\sqrt{p}\xi)\right], \tag{5.21}$$

profile of Eq. (5.21) is shown in Figure 5.4, for $\beta = 1/4,\ D = 1/2,\ k = 1/2,\ p = 1$ and $v(t) = \sin(2t)$.

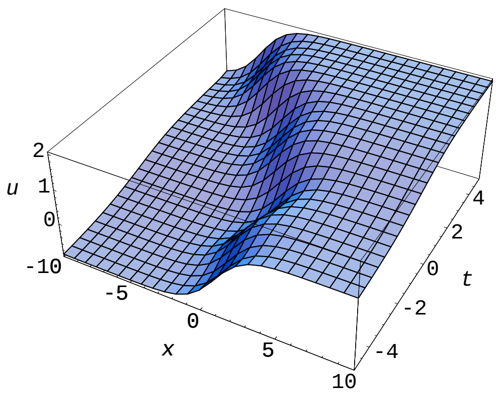

Figure 5.4: Typical form of u(x,t), Eq. (5.21) for values mentioned in the text.

5.4 Conclusion

In this chapter, we have studied the NLRD equations with time dependent coefficients. By using the auxiliary equation method, we have obtained the exact solitary wave solutions for these equations. We have found that, convective term coefficient can be any arbitrary function of time which affects the velocity of wave, where as reaction term coefficient can have only tanh time dependent term. Here, we have shown the variation of $u(x,t)$ by assuming $v(t)$ as any one of the sine, exponential and tanh functions. The approach applied in this work may be employed in further works to obtain new solutions for other types of nonlinear partial differential equations with time dependent variable coefficients. Most of the work presented here has appeared in Ref. [38].

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Chapter 6

Summary and conclusions

The study of nonlinear dynamics has been an active area of research since the discovery of solitary waves and solitons. In Chapter 1, we had introduced the concept of solitary waves and solitons. Solitary waves had the characteristic features of non-singularity, localization and stability. Those solitary waves which are stable against their mutual collisions are termed as Solitons. Mathematical modelling of many physical systems lead to nonlinear evolution equations. There are numerous examples of NLEEs like KdV equation, sine-Gordon equation, nonlinear Schrodinger equation, reaction-diffusion models etc. which had solitary waves and soliton like solutions. In the methodology section, we gave a brief description of various methods to find solutions of nonlinear systems such as the algebraic method, method of ansatz, factorization method, method of quadratures, Painlevé analysis, auxiliary equation method etc.

In Chapter 2, we study the higher order NLSE with cubic-quintic nonlinearities and self steepening and self frequency shift terms which is required for transmission of ultra short pulses through optical fibers. Propagating soliton like solutions were obtained and it was observed that these chirped pulses may prove useful in compression and amplification of optical pulses.

Chapter 3 is devoted to Factorization method. It is a very recent method developed by Rosu and his collaborators to solve second order nonlinear evolution equations. The basic idea underlying this method is to factorize a second order nonlinear equation into two first order equations and thus the complexity of solving a second order nonlinear equation reduces to solving a first order equation.

We had employed the factorization scheme to obtain particular solution to Convective Fisher equation and extended the factorization technique for the driven systems and implicit solution of Duffing-Van der Pol oscillator driven by constant force was obtained. We also obtained Riccati generalization of driven Convective Fisher equation.

We can conclude that not only Factorization Method reduces the complexity of solving second order NLEEs but it can be extended in certain directions such as for driven systems and higher order nonlinear evolution equations to obtain interesting solutions.

In chapter 4, we studied the nonlinear dynamics of DNA, for longitudinal and transverse motions, in the framework of the microscopic model of Peyrard and Bishop. The coupled nonlinear partial differential equations for dynamics of DNA model have been solved for solitary wave solution which is further generalized using Riccati parameterized factorization method.

When inhomogeneous media and non uniformities of the boundaries are taken into account in various physical situations, the variable coefficient NLEE's prove to be more powerful and realistic. Therefore, in chapter 5, we study nonlinear reaction diffusion equations with variable coefficients and obtain exact solitary wave solutions for them.

List of publications

Papers in refereed journals

1. **Alka**, A. Goyal, R. Gupta, C. N. Kumar and T. S. Raju, Chirped femtosecond solitons and double-kink solitons in the cubic-quintic nonlinear Schrödinger equation with self-steepening and self-frequency shift, ***Physical Review A*** 84 (2011) 063830.

2. **W. Alka**, A. Goyal and C. N. Kumar, Nonlinear dynamics of DNA - Riccati generalized solitary wave solutions, ***Physics Letters A*** 375 (2011) 480.

3. A. Goyal, **Alka**, R. Gupta and C. N. Kumar, Solitary wave solutions for Burgers-Fisher type equations with variable coefficients, ***World Academy of Science Engineering and Technology*** 60 (2011) 1742.

4. A. Goyal, **Alka**, T. S. Raju and C. N. Kumar, Lorentzian-type soliton solutions of ac-driven complex Ginzburg-Landau equation, ***Applied Mathematics and Computation*** 218 (2012) 11931.

Papers in conferences and workshops

1. A. Goyal, **Alka** and C. N. Kumar, Dynamics of a nonlinear model for DNA, 5^{th} ***Chandigarh Science Congress***, Panjab University, Feb. 26-28, 2011.

2. A. Goyal, **Alka** and C. N. Kumar, Solitary wave solutions of nonlinear reaction-diffusion equations with variable coefficients, 6^{th} ***National Conference on Nonlinear Systems and Dynamics***, Bharathidasan University, Jan. 27-30, 2011.

3. A. Goyal, **Alka** and C. N. Kumar, Nonlinear dynamics of elastic roads and application to DNA, 55^{th} ***Congress of Indian Society of Theoretical and Applied Mechanics***, NIT Hamirpur, Dec. 18-21, 2010.

4. **Alka** and C. N. Kumar, Solitary waves in optical media, 2^{nd} ***Chandigarh Science Congress***, Panjab University, March 14-15, 2008.

5. **Alka** and C. N. Kumar, Solitary wave solutions of a class of nonlinear evolution equations, 2^{nd} ***National Workshop on Techniques in Applied Mathematics***, Calcutta University, June 20-28, 2006.

Selected Reprints

Chirped femtosecond solitons and double-kink solitons in the cubic-quintic nonlinear Schrödinger equation with self-steepening and self-frequency shift

Alka, Amit Goyal, Rama Gupta, and C. N. Kumar
Department of Physics, Panjab University, Chandigarh 160 014, India

Thokala Soloman Raju*
Department of Physics, Karunya University, Coimbatore 641 114, India
(Received 1 October 2011; published 13 December 2011)

We demonstrate that the competing cubic-quintic nonlinearity induces propagating solitonlike dark(bright) solitons and double-kink solitons in the nonlinear Schrödinger equation with self-steepening and self-frequency shift. Parameter domains are delineated in which these optical solitons exist. Also, fractional-transform solitons are explored for this model. It is shown that the nonlinear chirp associated with each of these optical pulses is directly proportional to the intensity of the wave and saturates at some finite value as the retarded time approaches its asymptotic value. We further show that the amplitude of the chirping can be controlled by varying the self-steepening term and self-frequency shift.

DOI: 10.1103/PhysRevA.84.063830 PACS number(s): 42.81.Dp, 42.65.Tg

I. INTRODUCTION

The nonlinear Schrödinger equation (NLSE) in its many versions has various applications in different fields such as nonlinear optics [1], Bose-Einstein condensates [2], and biomolecular dynamics [3]. In nonlinear optics, the NLSE describes the dynamics of picosecond pulses that propagate in nonlinear media due to the delicate balance between group-velocity dispersion (GVD) and Kerr nonlinearity. However, over the past several years, ultrashort (femtosecond) pulses have been extensively studied due to their wide applications in many different areas such as ultrahigh-bit-rate optical communication systems, ultrafast physical processes, infrared time-resolved spectroscopy, and optical sampling systems [4]. To produce ultrashort pulses, the intensity of the incident light field increases, which leads to non-Kerr nonlinearities, changing the physical feature of the system. The dynamics of such systems should be described by the NLSE with higher-order terms such as third-order dispersion, self-steepening, and self-frequency shift [5,6]. Moreover, in some physical situations cubic-quintic nonlinear terms arise [7,8], due to non-Kerr nonlinearities, from a nonlinear correction to the refractive index of a medium. In general, unlike the NLSE, these models with non-Kerr effects are not completely integrable and cannot be solved exactly by the inverse scattering transform method [9]. Hence, they do not have soliton solutions; however, they do have solitary-wave solutions, which are often called solitons.

The effect of third-order dispersion is significant for femtosecond pulses when the GVD is close to zero. However, it can be neglected for pulses whose width is of the order of 100 fs or more, having power of the order of 1 W and GVD far away from zero [10]. However, the effects of self-steepening and self-frequency shift terms are still dominant and should be retained. Under these conditions, we have considered the higher-order NLSE with cubic-quintic nonlinearity of the form

$$i\psi_z + a_1\psi_{tt} + a_2|\psi|^2\psi + a_3|\psi|^4\psi + ia_4(|\psi|^2\psi)_t + ia_5\psi(|\psi|^2)_t = 0, \quad (1)$$

where $\psi(z,t)$ is the complex envelope of the electric field, a_1 is the parameter of GVD, a_2 and a_3 represent cubic and quintic nonlinearities, respectively, a_4 is the self-steepening coefficient, and a_5 is the self-frequency shift coefficient. For Eq. (1), many restrictive special solutions of the bright and dark types have been obtained [11,12]. Scalora *et al.* [13] used the model in Eq. (1), for $a_5 = 0$, to describe pulse propagation in a negative-index material, where the sign of GVD can be positive or negative.

Much of the work has been done on chirped pulses because of their application in pulse compression or amplification and thus they are particularly useful in the design of fiber-optic amplifiers, optical pulse compressors, and solitary-wave-based communications links [14,15]. The pulse with linear chirp and a hyperbolic-secant-amplitude profile was investigated numerically by Hmurcik and Kaup [16]. Subsequently, many authors have reported the existence of chirped solitonlike solutions [15,17,18]. One of the present authors solved Eq. (1) for $a_3 = 0$ and obtained solitonlike solutions with nonlinear chirp [10,19]. In this paper we consider the effect of quintic non-Kerr nonlinearity and obtain soliton solutions with a different form of chirping. We find that for certain parameter conditions between quintic, self-steepening, and self-frequency shift terms, the solutions will resemble NLSE solitons with velocity selection. We also report herein the existence of double-kink-type solitons with nonlinear chirp for Eq. (1). In all these cases, chirping varies as directly proportional to the intensity of the wave and saturates at some finite value as $t \rightarrow \pm\infty$. Further, we show that the amplitude of chirping can be controlled by varying the self-steeping and self-frequency shift terms. It is also shown that for the same values of all parameters, the equation can have either

*soloman@karunya.edu

dark(bright) solitons or double-kink-type solitons, depending upon the velocity and other parameters of the wave.

II. CHIRPED SOLITONLIKE SOLUTIONS

Here we are interested in finding chirped solitonlike solutions of Eq. (1). Hence we choose the following form for the complex envelope traveling-wave solutions:

$$\psi(z,t) = \rho(\xi)e^{i[\chi(\xi)-kz]}, \tag{2}$$

where $\xi = t - uz$ is the traveling coordinate and ρ and χ are real functions of ξ. Here $u = 1/v$, with v the group velocity of the wave packet. The corresponding chirp is given by $\delta\omega(t,z) = -\frac{\partial}{\partial t}[\chi(\xi) - kz] = -\chi'(\xi)$. Now, substituting Eq. (2) in Eq. (1) and separating out the real and imaginary parts of the equation, we arrive at the coupled equations in ρ and χ,

$$k\rho + u\chi'\rho - a_1\chi'^2\rho + a_1\rho'' - a_4\chi'\rho^3 + a_2\rho^3 + a_3\rho^5 = 0 \tag{3}$$

and

$$-u\rho' + a_1\chi''\rho + 2a_1\chi'\rho' + (3a_4 + 2a_5)\rho^2\rho' = 0. \tag{4}$$

To solve these coupled equations, we choose the ansatz

$$\chi'(\xi) = \alpha\rho^2 + \beta. \tag{5}$$

Hence, chirping is given as $\delta\omega(t,z) = -(\alpha\rho^2 + \beta)$, where α and β denote the nonlinear and constant chirp parameters, respectively. Using this ansatz in Eq. (4), we get the relations

$$\alpha = -\frac{3a_4 + 2a_5}{4a_1}, \quad \beta = \frac{u}{2a_1}. \tag{6}$$

Hence, the value of the chirp parameter depends on different coefficients of the evolution equation (1) such as diffraction, self-steepening, and self-frequency shift. This means that the amplitude of chirping can be controlled by varying these coefficients. Now using Eqs. (5) and (6) in Eq. (3), we obtain

$$\rho'' + b_1\rho^5 + b_2\rho^3 + b_3\rho = 0, \tag{7}$$

where $b_1 = \frac{1}{16a_1^2}[16a_1a_3 - (2a_5 + 3a_4)(2a_5 - a_4)]$, $b_2 = \frac{1}{2a_1^2}(2a_1a_2 - ua_4)$, and $= b_3 = \frac{1}{4a_1^2}(4ka_1 + u^2)$.

This elliptic equation is known to admit a variety of solutions such as periodic, kink, and solitary-wave-type solutions. In general, all traveling-wave solutions of Eq. (7) can be expressed in a generic form by means of the Weierstrass $\wp$ function [20]. In this paper we report various localized solutions for different parameter conditions. It is interesting to note that if $b_1 = 0$, i.e., the quintic term is related to self-steepening and self-frequency shift terms, then Eq. (7) reduces to a cubic nonlinear equation that admits dark and bright solitons. For the case when $b_2 = 0$, it can be solved for localized solutions by using a fractional transformation. For $b_3 = 0$, we show that the equation has a Lorentzian-type solution. In the most general case, when all the coefficients have nonzero values, Eq. (7) can be mapped onto a ϕ^6 field equation to obtain double-kink-type [21] and bright- and dark-soliton solutions [22] of Eq. (1).

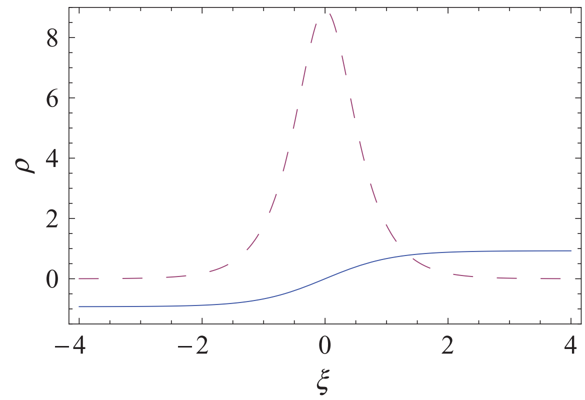

FIG. 1. (Color online) Amplitude profile for the (a) dark soliton (solid line) for $u = 4.1184$ and $k = 0$ and (b) bright soliton (dashed line) for $u = -30.1280$ and $k = -150.2856$.

In the following we delineate the parameter domains in which solitonlike solutions exist for this model. For example, when $b_1 = 0$ two interesting cases emerge that yield exact soliton solutions. (a) For $b_2 < 0$ and $b_3 > 0$, which implies $u > \frac{2a_1a_2}{a_4}$ and $k > \frac{-u^2}{4a_1}$, one obtains a dark-soliton solution of Eq. (1) of the form

$$\psi(z,t) = \sqrt{-\frac{b_3}{b_2}}\tanh\left(\sqrt{\frac{b_3}{2}}(t - uz)\right)e^{i[\chi(\xi)-kz]}. \tag{8}$$

The corresponding chirping is given by

$$\delta\omega(t,z) = \frac{\alpha b_3}{b_2}\tanh^2\left(\sqrt{\frac{b_3}{2}}\xi\right) - \beta. \tag{9}$$

(b) For $b_2 > 0$ and $b_3 < 0$, which implies $u < \frac{2a_1a_2}{a_4}$ and $k < \frac{-u^2}{4a_1}$, one can find a bright-soliton solution of the form

$$\psi(z,t) = \sqrt{-\frac{2b_3}{b_2}}\mathrm{sech}[\sqrt{-b_3}(t - uz)]e^{i[\chi(\xi)-kz]}, \tag{10}$$

for which the chirping will be

$$\delta\omega(t,z) = \frac{2\alpha b_3}{b_2}\mathrm{sech}^2(\sqrt{-b_3}\xi) - \beta. \tag{11}$$

Hence, the parametric condition $b_1 = 0$ implies that $16a_1a_3 = (2a_5 + 3a_4)(2a_5 - a_4)$ and the amplitude profile will be the

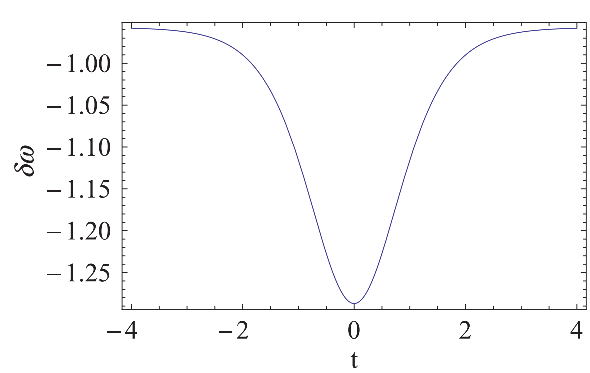

FIG. 2. (Color online) Chirping profile for the dark soliton plotted in Fig. 1.

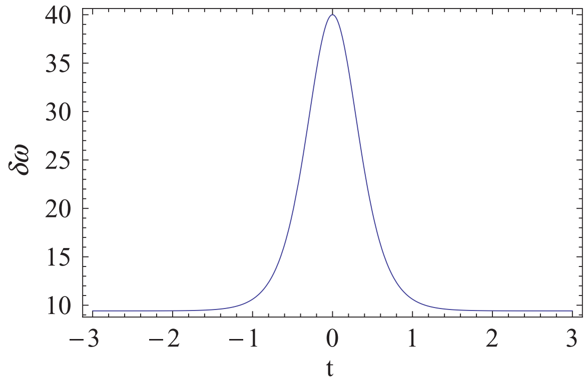

FIG. 3. (Color online) Chirping profile for the bright soliton plotted in Fig. 1.

same as for the NLSE, except the chirping will show nonlinear behavior. However, unlike for the NLSE, both dark and bright solitons exist in the normal and anomalous dispersion regimes. However, both soliton solutions have mutually exclusive velocity space. The amplitude profile of a typical dark and bright soliton is shown in Fig. 1, using the following values for the model parameters: $a_1 = 1.6001, a_2 = -2.6885, a_4 = 0.308\,14$, and $a_5 = 0.766\,04$. As $b_1 = 0$, the quintic-term coefficient is $a_3 = 0.1174$. The corresponding chirping for dark and bright solitons is shown in Figs. 2 and 3, respectively (for $z = 0$). It is clear from the figure that chirping for the dark soliton has a minimum at the center of the pulse, whereas for the bright soliton it has a maximum; however, for both cases it saturates at the same finite value as $t \to \pm\infty$.

III. CHIRPED FRACTIONAL-TRANSFORM SOLITONS

For the parametric condition $b_2 = 0$, we obtain very interesting chirped fractional-transform soliton. To accomplish this we now substitute $\rho^2 = y$ in Eq. (7), which can then be reduced to the following elliptic equation:

$$y'' + \tfrac{8}{3} b_1 y^3 + 4 b_3 y + c_0 = 0. \quad (12)$$

It is shown here that this elliptic equation connects to the well-known elliptic equation $f'' \pm af \pm bf^3 = 0$, where a and b are real, using a fractional transformation [23]

$$y(\xi) = \frac{A + Bf^2(\xi)}{1 + Df^2(\xi)}, \quad (13)$$

and we obtain the nontrivial Lorentzian-type solitons of Eq. (1).

Our main aim is to study the localized solutions: We consider the case where $f = \mathrm{cn}(\xi, m)$ with modulus parameter $m = 1$, which reduces $\mathrm{cn}(\xi)$ to $\mathrm{sech}(\xi)$. We can see that Eq. (13) connects $y(\xi)$ to the elliptic equation, provided $AD \neq B$, and the following conditions should be satisfied for the localized solution:

$$12 b_3 A + 8 b_1 A^3 + 3 c_0 = 0, \quad (14)$$

$$8 b_3 AD + 4 b_3 B + 4(B - AD) + 8 b_1 A^2 B + 3 c_0 D = 0, \quad (15)$$

$$4 b_3 AD^2 + 8 b_3 BD + 4(AD - B)D + 6(AD - B) + 8 b_1 AB^2 + 3 c_0 D^2 = 0, \quad (16)$$

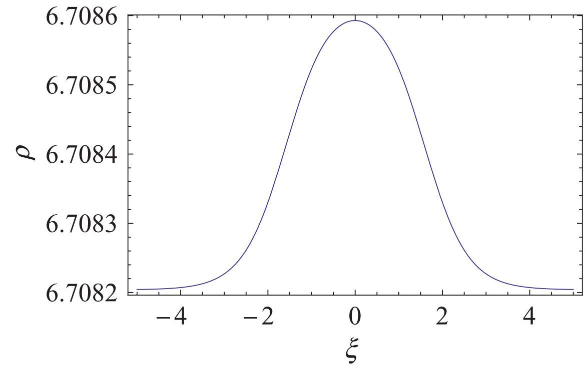

FIG. 4. (Color online) Typical amplitude profile for the soliton solution given by Eq. (19) for the values mentioned in the text.

$$12 b_3 BD^2 + 6(B - AD)D + 8 b_1 B^3 + 3 c_0 D^3 = 0. \quad (17)$$

From Eq. (15) we find that $D = \Gamma B$, where $\Gamma = \frac{4 + 4b_3 + 8b_1 A^2}{4A - 8b_3 A - 3c_0}$. Using this in Eq. (16), we determine B as

$$B = \frac{6(1 - A\Gamma)}{8 b_1 A + 4 b_3 \Gamma^2 A + 8 b_3 \Gamma + 4\Gamma(A\Gamma - 1) + 3 c_0 \Gamma^2}.$$

By substituting these expressions in Eqs. (14) and (17), we can determine A and c_0 for any given values of b_1 and b_3.

Thus, the localized solution is of the form

$$y(\xi) = \frac{A + B\,\mathrm{sech}^2(\xi)}{1 + D\,\mathrm{sech}^2(\xi)} \quad (18)$$

and $\rho(\xi)$ can be written as

$$\rho(\xi) = \sqrt{\frac{A + B\,\mathrm{sech}^2(\xi)}{1 + D\,\mathrm{sech}^2(\xi)}}. \quad (19)$$

The chirping takes the form

$$\delta\omega(t,z) = -\left[\alpha\left(\frac{A + B\,\mathrm{sech}^2(\xi)}{1 + D\,\mathrm{sech}^2(\xi)}\right) + \beta\right]. \quad (20)$$

The typical profiles for amplitude and chirping (for $z = 0$) are shown in Figs. 4 and 5, respectively, for $a_1 = 1.6001$, $a_2 = -2.6885$, $a_3 = 0.0260$, $a_4 = 0.308\,14$, $a_5 = 0.766\,04$, and $k = 0$. To make $b_2 = 0$, we set $u = -27.9215$.

In the following, we obtain yet another interesting algebraic soliton for the parametric condition $b_3 = 0$. In particular, for

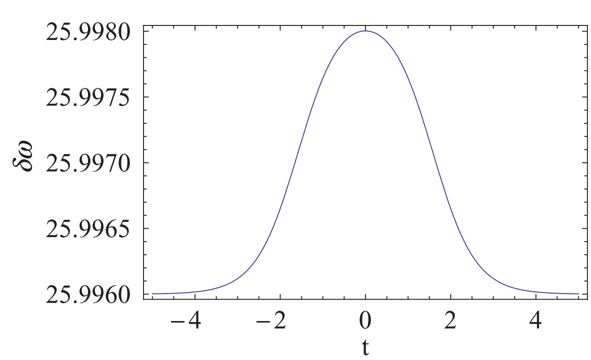

FIG. 5. (Color online) Chirping profile for the soliton solution plotted in Fig. 4.

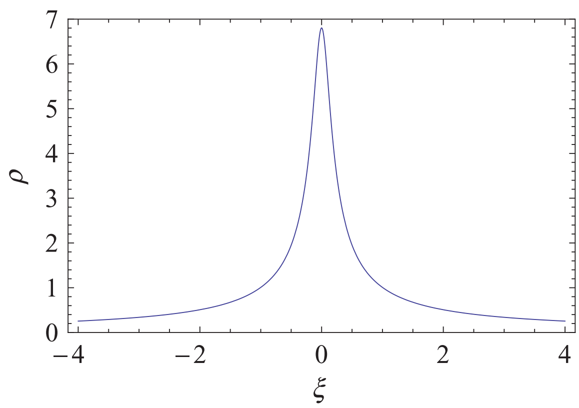

FIG. 6. (Color online) Typical amplitude profile for the soliton solution given by Eq. (21) for the values mentioned in the text.

$b_2 < 0$ and $b_1 > 0$, the solution of Eq. (7) is of the following form:

$$\rho(\xi) = \frac{1}{\sqrt{M + N\xi^2}}, \tag{21}$$

where $M = \frac{-2b_1}{3b_2}, N = \frac{-b_2}{2}$, and the chirping is given by

$$\delta\omega(t,z) = -\left(\frac{\alpha}{M + N\xi^2} + \beta\right). \tag{22}$$

For this case, the typical profiles for the amplitude and chirping (for $z = 0$) are shown in Figs. 6 and 7, respectively, for $a_1 = 1.6001$, $a_2 = -2.6885$, $a_3 = 0.2174$, $a_4 = 0.308\,14$, $a_5 = 0.766\,04$, and $u = 4.1185$. For $b_3 = 0$, we set $k = -121.8064$.

IV. CHIRPED DOUBLE-KINK AND BRIGHT(DARK) SOLITONS

We now demonstrate the existence of double-kink solitons and bright(dark) solitons when all the parameters in Eq. (7) are nonzero, i.e., $b_1 \neq 0$, $b_2 \neq 0$, and $b_3 \neq 0$. For the general case, Eq. (7) can be solved for double-kink-type (usually called two-kink) soliton solutions of the form [21]

$$\rho(\xi) = \frac{p\sinh(q\xi)}{\sqrt{\epsilon + \sinh^2(q\xi)}}, \tag{23}$$

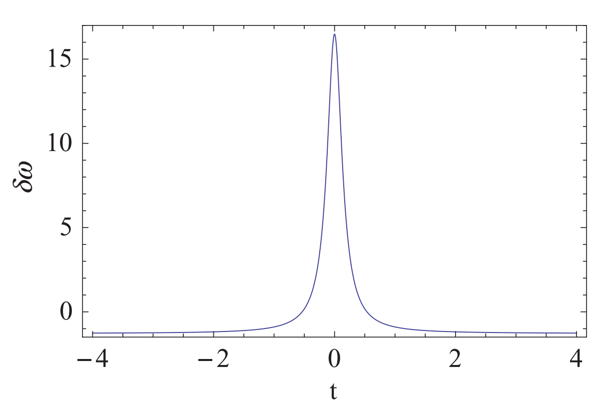

FIG. 7. (Color online) Chirping profile for the soliton solution plotted in Fig. 6.

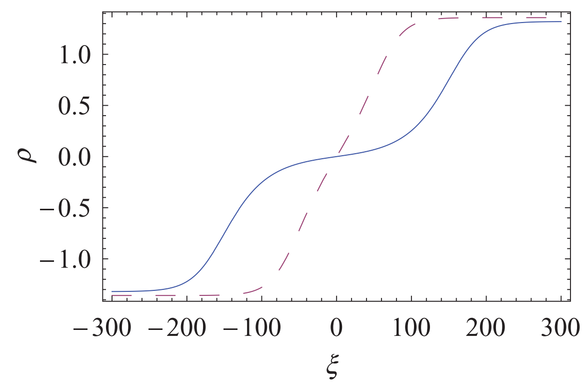

FIG. 8. (Color online) Typical amplitude profile of the soliton solution in Eq. (23) for different values of ϵ: $\epsilon = 1000$ for $p = 1.3204, q = 0.0252$, and $k = -141.911$ (solid line) and $\epsilon = 10$ for $p = 1.3584, q = 0.0287$, and $k = -141.899$ (dashed line).

where $b_1 = -\frac{3q}{p}(\frac{\epsilon-1}{\epsilon}), b_2 = 2pq(\frac{2\epsilon-3}{\epsilon})$, and $b_3 = -p^3q(\frac{\epsilon-3}{\epsilon})$. For this case, the chirping can be written as

$$\delta\omega(t,z) = -\left(\frac{\alpha p^2\sinh^2(q\xi)}{\epsilon + \sinh^2(q\xi)} + \beta\right). \tag{24}$$

The amplitude profile of the soliton solution for different values of ϵ is shown in Fig. 8 for $a_1 = 1.6001$, $a_2 = -2.6885$, $a_3 = 0.0260$, $a_4 = 0.308\,14$, $a_5 = 0.766\,04$, and $u = -30.1280$. The interesting double-kink feature of the solution given by Eq. (23) exists only for sufficiently large values of ϵ. One can also point out that as the value of ϵ changes, it effects only the width of the wave, but the amplitude of the wave remains the same. Chirping for the solution is shown in Fig. 9 (for $z = 0$), which has a minimum at the center of the pulse and saturates at the same finite value as $t \to \pm\infty$.

It is interesting to note that for $b_3 < 0$ and $b_2 > 0$, Eq. (7) has both bright and dark solitons depending on the value of b_1 [19]. The explicit solutions and corresponding chirping are given below.

If $b_1 < |\frac{3b_2^2}{16b_3}|$ then Eq. (7) has a bright-soliton-type solution, which is given as

$$\rho(\xi) = \frac{p}{\sqrt{1 + r\cosh q\xi}}, \tag{25}$$

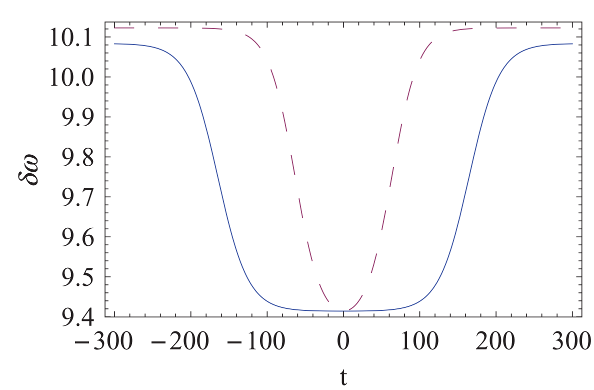

FIG. 9. (Color online) Chirping profile for the soliton solutions plotted in Fig. 8 for $\epsilon = 1000$ (solid line) and $\epsilon = 10$ (dashed line).

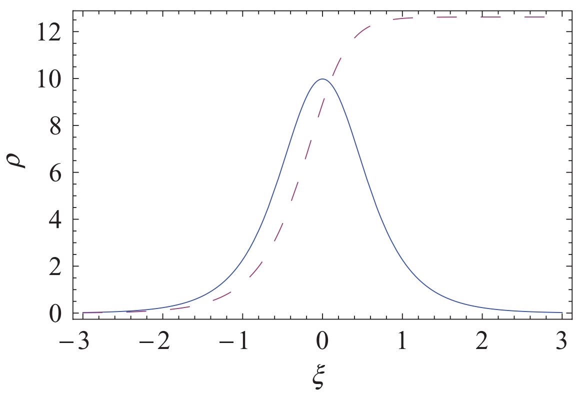

FIG. 10. (Color online) Amplitude profile of the soliton solutions in Eqs. (25) and (27) for $u = -30.1280$ and $k = -150.2856$: the bright soliton for $a_3 = 0.1168$ (solid line) and the dark soliton for $a_3 = 0.1164$ (dashed line).

where $p^2 = -\frac{4b_3}{b_2}, q^2 = -4b_3$, and $r^2 = 1 - \frac{16b_1b_3}{3b_2^2}$. The corresponding chirping is

$$\delta\omega(t,z) = -\left(\frac{\alpha p^2}{1 + r\cosh q\xi} + \beta\right). \tag{26}$$

If $b_1 = \frac{3b_2^2}{16b_3}$, then the solution of Eq. (7) will be of the dark-soliton type given by

$$\rho(\xi) = \pm p\sqrt{1 \pm \tanh q\xi}, \tag{27}$$

where $p^2 = -\frac{2b_3}{b_2}$ and $q^2 = -b_3$. For this case, the chirping is given as

$$\delta\omega(t,z) = -[\alpha p^2(1 \pm \tanh q\xi) + \beta]. \tag{28}$$

In Fig. 10 the amplitude profile of typical bright and dark solitons is shown for $a_1 = 1.6001$, $a_2 = -2.6885$, $a_4 = 0.308\,14$, and $a_5 = 0.766\,04$. It is interesting to note that Eq. (7) has bright and dark solitons depending on the value of the quintic-term coefficient, i.e., a_3, as shown in figure. It is shown that chirping (for $z = 0$) is also different in both cases: For a bright soliton chirping has maxima at the center of the pulse, which saturates at the same finite value (see Fig. 11), whereas for a dark soliton it saturates to different finite values as $t \to \pm\infty$. Hence, Eq. (7) has bright(dark) soliton

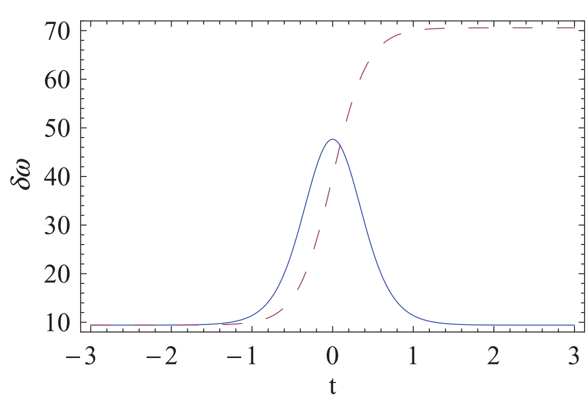

FIG. 11. (Color online) Chirping profile for the solitons plotted in Fig. 10: the bright soliton (solid line) and the dark soliton (dashed line).

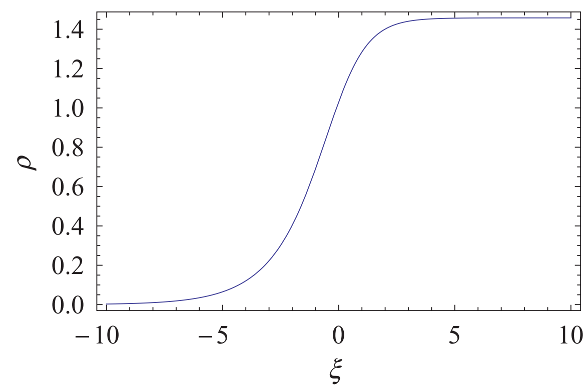

FIG. 12. (Color online) Amplitude profile of the soliton solution in Eq. (33) for $\gamma = 2.1245, \Delta = 0.2927$, and $k = -142.438$.

and double-kink-type soliton solutions for the same model parameters but for a different velocity selection and other parameters of the wave.

V. CHIRPED KINK SOLITONS FOR PURELY IMAGINARY a_5

Here we elucidate a more physically interesting case for which a_5 is imaginary in Eq. (1). Thus, for $a_5 \to ia_5$, Eqs. (3) and (4) read

$$k\rho + u\chi'\rho - a_1\chi'^2\rho + a_1\rho'' - a_4\chi'\rho^3 \\ +a_2\rho^3 + a_3\rho^5 - 2a_5\rho^2\rho' = 0 \tag{29}$$

and

$$-u\rho' + a_1\chi''\rho + 2a_1\chi'\rho' + 3a_4\rho^2\rho' = 0. \tag{30}$$

Substituting Eq. (5) in Eq. (30), we obtain

$$\alpha = -\frac{3a_4}{4a_1}, \quad \beta = \frac{u}{2a_1}. \tag{31}$$

Now using Eqs. (5) and (31) in Eq. (29), we obtain

$$\rho'' + b_1\rho^5 + b_2\rho^3 + b_3\rho + b_4\rho^2\rho' = 0, \tag{32}$$

where $b_1 = \frac{1}{16a_1^2}(16a_1a_3 + 3a_4^2), b_2 = \frac{1}{2a_1^2}(2a_1a_2 - ua_4), b_3 = \frac{1}{4a_1^2}(4ka_1 + u^2)$, and $b_4 = -\frac{2a_5}{a_1}$. Equation (32) can be solved for a kink-type soliton solution of the form

$$\rho(\xi) = \sqrt{\frac{\Delta}{2}}\sqrt{1 + \tanh(\gamma\Delta\xi)}, \tag{33}$$

where $\gamma = \frac{b_4 \pm \sqrt{b_4^2 - 12b_1}}{6}, \Delta = \frac{b_2}{4\gamma^2 - b_4\gamma}$, and b_3 satisfies the condition that $b_3 = -\gamma^2\Delta^2$. The chirping is given by

$$\delta\omega(t,z) = -\left[\frac{\alpha\Delta}{2}[1 + \tanh(\gamma\Delta\xi)] + \beta\right]. \tag{34}$$

The amplitude profile of the soliton solution for different values of ϵ is shown in Fig. 12 for $a_1 = 1.6001$, $a_2 = -2.6885$, $a_3 = 0.0260$, $a_4 = 0.308\,14$, $a_5 = 0.766\,04$, and $u = -30.1280$. Chirping for this kink-type solution (depicted in Fig. 13) saturates to different finite values as $t \to \pm\infty$.

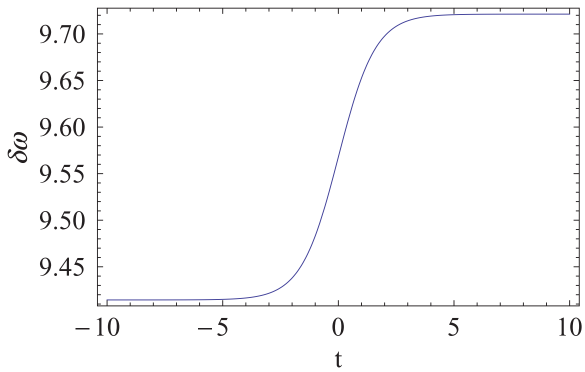

FIG. 13. (Color online) Chirping profile for the soliton solution plotted in Fig. 12.

VI. CONCLUSION

We would like to point out that the present work is a natural but significant generalization of Ref. [10] by considering the effect of competing cubic-quintic nonlinearity on the ensuing optical solitons in the higher-order nonlinear Schrödinger equation. We have demonstrated that the competing cubic-quintic nonlinearity induces propagating solitonlike dark(bright) solitons and double-kink solitons in the nonlinear Schrödinger equation with self-steepening and self-frequency shift. Parameter domains were delineated in which these optical solitons exist. In addition, fractional transform solitons were explored for this model. It was shown that the nonlinear chirp associated with each of these optical solitons is directly proportional to the intensity of the wave and saturates at some finite value as the retarded time approaches its asymptotic value. We have further shown that the amplitude of the chirping can be controlled by varying the self-steepening term and self-frequency shift. These optical solitons have nontrivial phase chirping that varies as a function of intensity and are different from that in Ref. [11], where the solution had a trivial phase. We hope that these chirped femtosecond solitons and double-kink solitons may be launched in long-distance telecommunication networks involving higher-order nonlinearities of the fiber.

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Nonlinear dynamics of DNA – Riccati generalized solitary wave solutions

W. Alka, Amit Goyal, C. Nagaraja Kumar *

Department of Physics, Panjab University, Chandigarh-160014, India

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ABSTRACT

We study the nonlinear dynamics of DNA, for longitudinal and transverse motions, in the framework of the microscopic model of Peyrard and Bishop. The coupled nonlinear partial differential equations for dynamics of DNA model, which consists of two long elastic homogeneous strands connected with each other by an elastic membrane, have been solved for solitary wave solution which is further generalized using Riccati parameterized factorization method.

1. Introduction

Deoxyribonucleic acid (DNA) molecule encodes the information that organisms need to live and reproduce themselves. It is an attractive system for nonlinear science because its properties can be probed very accurately by experiments that combine the methods of physics with biological tools [1]. DNA structure has been extensively studied during the last decades. It consists of a pair of molecules, organized as strands and joined by hydrogen as well as covalent bonds. The diameter of the double stranded DNA is 2 nm indeed, but the length could be much longer, usually. In 1953, Watson and Crick [2] firstly discovered the double helix structure of DNA, but still it is really difficult to relate all its characteristics to a specific mathematical model due to its complex structure and presence of various motions (for example, the longitudinal, transverse and torsional motions) [3]. However different motions of DNA are present on quiet different time scales, which helps to model few of these which dominate in the given range of time scale.

Theoretical studies of the nonlinear properties of DNA have been stimulated by the pioneer works of Davydov [4]. His idea was firstly applied to DNA by Englander and coauthors in 1980 [5], who studied the dynamics of DNA open states taking into account only the rotational motion of nitrogen bases. Yomosa [6] has developed this idea further and has proposed a dynamics plane-base rotator model that was later improved by Takeno and Homma [7], in which attention was paid to the degree of freedom characterizing base rotations in the plane perpendicular to the helical axis around the backbone structure. Peyrard and Bishop [8] studied the process of denaturation in which only the transverse motions of bases along the hydrogen bond was taken into account. Further, Muto and coauthors [9] suggested that two types of internal motions made the main contribution to the DNA denaturation process: transverse motions along the hydrogen bond and longitudinal motions along the backbone direction. In recent years, further improvements were made in the model [10–14] to study dynamics of DNA for different motions and obtained solitary wave type solutions. These localized nonlinear excitations explain the conformation transition [15,16], the long range interaction of kink solitons in the double chain [17,18], the regulation of transcription [17,19], denaturation [8,9] and charge transport in terms of polarons and bubbles [20].

In this Letter, we have studied the nonlinear dynamics of DNA for longitudinal and transverse modes and obtained a new class of solitary wave solutions. The coupled nonlinear partial differential equations, describing the nonlinear dynamics of DNA, can be reduced to single ordinary differential equation (ODE) in the travelling wave frame which admits kink and anti-kink solutions [21]. It is known in the context of quantum mechanics [22,23] and in general [24], that a given second order ODE can be solved by factorizing it into product of two first order operators and its Riccati generalized solution [25] can be found for a particular form of factorization. Using this fact, we have obtained one particular family of solitary kink-type solutions for different values of Riccati parameter.

2. Model of the DNA and equations

In this Letter, we consider the model used by Kong et al. [26] which is based on the lattice model of Peyrard and Bishop [8]. In

* Corresponding author.
E-mail address: cnkumar@pu.ac.in (C. Nagaraja Kumar).

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this model, DNA molecule is supposed to consist of two long elastic strands (or rods) which represents two polynucleotide chains of the DNA molecule, connected to each other by an elastic membrane representing the hydrogen bonds between the pairs of bases in the two chains. We assume a homogeneous DNA molecule, therefore both strands have same mass density. In order to make the Letter self-contained, we sketch the essential steps of [26] and to simplify the calculations, we shall make two assumptions. First, we neglect the helical structure of DNA. So, instead of the double helix we shall consider two parallel strands, each having the form of a straight line. Second, we consider the longitudinal and transverse motions of DNA strands, and neglect the torsional motion. Therefore, the model includes four degrees of freedom, u_1, v_1 and u_2, v_2, for the two strands respectively. The u_1 and u_2 represent respectively the longitudinal displacements of the top strand and the bottom strand i.e. the displacement of the bases from their equilibrium positions along the direction of the phosphodiester bridge that connects the two bases of the same strand; v_1 and v_2 denote respectively the transverse displacements of the top strand and the bottom strand i.e. the displacement of the bases from their equilibrium positions along the direction of the hydrogen bonds that connect the two bases of the base pair.

The Hamiltonian of such a model can be written as

$$H = T + V_1 + V_2, \tag{1}$$

where T is the kinetic energy of the elastic strands, V_1 is the potential energy of the elastic strands and V_2 is the potential energy of the elastic membrane. Here

$$T = \int \frac{1}{2}\rho\sigma\left(\dot{u_1}^2 + \dot{u_2}^2 + \dot{v_1}^2 + \dot{v_2}^2\right)dx, \tag{2}$$

$$V_1 = \int \frac{1}{2}Y\sigma\left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial x}\right)^2\right] + \frac{1}{2}F\sigma\left[\left(\frac{\partial v_1}{\partial x}\right)^2 + \left(\frac{\partial v_2}{\partial x}\right)^2\right]dx, \tag{3}$$

and

$$V_2 = \int \frac{1}{2}\mu\big(\Delta l(x)\big)^2\,dx, \tag{4}$$

where ρ, σ, Y and F denote respectively the mass density, the area of transverse cross-section, the Young's modulus and the tension density of each strand; μ is the rigidity of the elastic membrane and $\Delta l(x)$ is the stretching of the elastic membrane at x due to longitudinal vibrations and is given by

$$\Delta l = \sqrt{(h + v_1 - v_2)^2 + (u_2 - u_1)^2} - l_0, \tag{5}$$

where h is the distance between the two strands and l_0 is the height of the membrane in the equilibrium position.

The dynamical equations associated with Hamiltonian can be written as

$$\begin{aligned}
\sigma\left[\rho\frac{\partial^2 u_1}{\partial t^2} - Y\frac{\partial^2 u_1}{\partial x^2}\right] &= \mu\frac{\Delta l}{l_0 + \Delta l}(u_2 - u_1),\\
\sigma\left[\rho\frac{\partial^2 u_2}{\partial t^2} - Y\frac{\partial^2 u_2}{\partial x^2}\right] &= \mu\frac{\Delta l}{l_0 + \Delta l}(u_1 - u_2),\\
\sigma\left[\rho\frac{\partial^2 v_1}{\partial t^2} - F\frac{\partial^2 v_1}{\partial x^2}\right] &= \mu\frac{\Delta l}{l_0 + \Delta l}(v_2 - v_1 - h),\\
\sigma\left[\rho\frac{\partial^2 v_2}{\partial t^2} - F\frac{\partial^2 v_2}{\partial x^2}\right] &= \mu\frac{\Delta l}{l_0 + \Delta l}(v_1 - v_2 + h).
\end{aligned} \tag{6}$$

From Eq. (5), we have

$$\frac{\Delta l}{l_0 + \Delta l} = \frac{\sqrt{(h + v_1 - v_2)^2 + (u_2 - u_1)^2} - l_0}{\sqrt{(h + v_1 - v_2)^2 + (u_2 - u_1)^2}}. \tag{7}$$

Assuming $|u_1 - u_2| \ll h$, $|v_1 - v_2| \ll h$ and neglecting higher order (> 2) terms, Eq. (7) can be written as

$$\begin{aligned}
\frac{\Delta l}{l_0 + \Delta l} &= R(u_1, u_2, v_1, v_2)\\
&= 1 - \frac{l_0}{h} + \frac{l_0}{h^2}(v_1 - v_2) - \frac{l_0}{h^3}(v_1 - v_2)^2\\
&\quad + \frac{l_0}{2h^3}(u_2 - u_1)^2.
\end{aligned} \tag{8}$$

Then Eq. (6) reduces to

$$\begin{aligned}
\sigma\left[\rho\frac{\partial^2 u_1}{\partial t^2} - Y\frac{\partial^2 u_1}{\partial x^2}\right] &= \mu R(u_1, u_2, v_1, v_2)(u_2 - u_1),\\
\sigma\left[\rho\frac{\partial^2 u_2}{\partial t^2} - Y\frac{\partial^2 u_2}{\partial x^2}\right] &= \mu R(u_1, u_2, v_1, v_2)(u_1 - u_2),\\
\sigma\left[\rho\frac{\partial^2 v_1}{\partial t^2} - F\frac{\partial^2 v_1}{\partial x^2}\right] &= \mu R(u_1, u_2, v_1, v_2)(v_2 - v_1 - h),\\
\sigma\left[\rho\frac{\partial^2 v_2}{\partial t^2} - F\frac{\partial^2 v_2}{\partial x^2}\right] &= \mu R(u_1, u_2, v_1, v_2)(v_1 - v_2 + h).
\end{aligned} \tag{9}$$

3. Solitary wave solutions

We can separate Eq. (9) into in-phase and out-of-phase motion by changing the variables as $u_+ = \frac{(u_1+u_2)}{\sqrt{2}}$ and $v_+ = \frac{(v_1+v_2)}{\sqrt{2}}$ for in-phase motion and $u_- = \frac{(u_2-u_1)}{\sqrt{2}}$ and $v_- = \frac{(v_2-v_1)}{\sqrt{2}}$ for out-of-phase motion. As out-of-phase motion stretches the hydrogen bond [8, 27], we will consider only out-of-phase motion for which equation of motion can be rewritten from Eq. (9) as

$$\begin{aligned}
\frac{\partial^2 u_-}{\partial t^2} - c_1^2\frac{\partial^2 u_-}{\partial x^2} &= \lambda_1 u_- + \gamma_1 u_- v_- + \mu_1 u_-^3 + \beta_1 u_- v_-^2,\\
\frac{\partial^2 v_-}{\partial t^2} - c_2^2\frac{\partial^2 v_-}{\partial x^2} &= \lambda_2 v_- + \gamma_2 u_-^2 + \mu_2 u_-^2 v_- + \beta_2 v_-^3 + c_0,
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
&c_1 = \pm\sqrt{\frac{Y}{\rho}}; \qquad c_2 = \pm\sqrt{\frac{F}{\rho}}; \qquad \lambda_1 = \frac{-2\mu}{\rho\sigma h}(h - l_0);\\
&\lambda_2 = \frac{-2\mu}{\rho\sigma}; \qquad \gamma_1 = 2\gamma_2 = \frac{2\sqrt{2}\mu l_0}{\rho\sigma h^2};\\
&\mu_1 = \mu_2 = \frac{-2\mu l_0}{\rho\sigma h^3}; \qquad \beta_1 = \beta_2 = \frac{4\mu l_0}{\rho\sigma h^3};\\
&c_0 = \frac{\sqrt{2}\mu(h - l_0)}{\rho\sigma}.
\end{aligned} \tag{11}$$

Introducing the transformation $v_- = au_- + b$ (a and b are constants), in Eq. (10) gives us

$$\begin{aligned}
\frac{\partial^2 u_-}{\partial t^2} - c_1^2\frac{\partial^2 u_-}{\partial x^2} &= u_-^3\left(\mu_1 + \beta_1 a^2\right) + u_-^2(2\beta_1 ab + a\gamma_1)\\
&\quad + u_-\left(\lambda_1 + b\gamma_1 + \beta_1 b^2\right),
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\frac{\partial^2 u_-}{\partial t^2} - c_2^2\frac{\partial^2 u_-}{\partial x^2}\\
&\quad = u_-^3\left(\mu_2 + \beta_2 a^2\right) + u_-^2\left(\frac{\gamma_2}{a} + \frac{\mu_2 b}{a} + 3\beta_2 ab\right)\\
&\qquad + u_-\left(\lambda_2 + 3\beta_2 b^2\right) + \frac{\lambda_2 b}{a} + \frac{\beta_2 b^3}{a} + \frac{c_0}{a}.
\end{aligned} \tag{13}$$

Comparing Eqs. (12) and (13) and by using Eq. (11), we have $b = \frac{h}{\sqrt{2}}$ and $F = Y$. Renaming u_- as ϕ, Eqs. (12) and (13) can be written as

$$\frac{\partial^2\phi}{\partial t^2} - c_1^2\frac{\partial^2\phi}{\partial x^2} = A\phi^3 + B\phi^2 + C\phi + D, \tag{14}$$

where

$$A = \left(\frac{-2\alpha}{h^3} + \frac{4a^2\alpha}{h^3}\right); \qquad B = \frac{6\sqrt{2}a\alpha}{h^2};$$
$$C = \left(\frac{-2\alpha}{l_0} + \frac{6\alpha}{h}\right); \qquad D = 0; \quad \text{with } \alpha = \frac{\mu l_0}{\rho\sigma}. \tag{15}$$

In travelling frame, Eq. (14) can be written as

$$\phi_{\xi\xi} = A\phi^3 + B\phi^2 + C\phi, \tag{16}$$

where $\xi = \frac{x}{c_1} - \sqrt{2}\,t$. In order to solve Eq. (16), let's assume

$$\phi = M\left(\frac{y'}{y}\right) + N, \tag{17}$$

where M and N are constants, and y satisfies the elliptic equation

$$y'' = \left(p + qy^2\right)y. \tag{18}$$

From Eqs. (17) and (18), we get the following equation

$$\phi'' = \frac{2}{M^2}\phi^3 - \frac{6N}{M^2}\phi^2 + \left(\frac{6N^2}{M^2} - 2p\right)\phi + \left(2Np - \frac{2N^3}{M^2}\right). \tag{19}$$

Comparing Eqs. (16) and (19), we have

$$M = \pm\sqrt{\frac{h^3}{\alpha(2a^2-1)}}; \qquad N = -\frac{\sqrt{2}ha}{(2a^2-1)};$$
$$p = \frac{N^2}{M^2} = \frac{2a^2\alpha}{h(2a^2-1)}; \quad \text{with } a^2 = \frac{3l_0 - h}{2(l_0 - h)}, \tag{20}$$

as $p = \frac{N^2}{M^2} \Rightarrow p > 0$ and $a^2 > \frac{1}{2}$.

From Eq. (18), $y = R\,\mathrm{sech}(m\xi)$, where $R^2 = -\frac{2p}{q}$ and $m^2 = p$. Then Eq. (16) has the following exact solitary wave solution

$$u_- \equiv \phi = M\left(\frac{y'}{y}\right) + N = -M\sqrt{p}\,\tanh(\sqrt{p}\xi) + N. \tag{21}$$

Substituting for M, N and p from Eq. (20), we have

$$u_- = \frac{-\sqrt{2}ah}{(2a^2-1)}\left[1 \pm \tanh\left(\sqrt{\frac{2a^2\mu l_0}{\rho\sigma h(2a^2-1)}}\xi\right)\right], \tag{22}$$

in which, the plus or minus sign stands for anti-kinks or kinks respectively, and $a^2 > \frac{1}{2}$ and $\xi = \frac{1}{c_1}(x - \sqrt{2}c_1 t)$ such that $c_1 = \pm\sqrt{\frac{Y}{\rho}}$. It is interesting to note that the amplitude and width of the wave is arbitrary modulo with $a^2 > \frac{1}{2}$.

4. Riccati generalization of solution

Rosu et al. [24] offers a simple way to solve a second order nonlinear differential equation by using factorization method. If a second order differential equation can be factorized to following form

$$\left[D - f_2(\phi)\right]\left[D - f_1(\phi)\right]\phi = 0, \tag{23}$$

then the particular solution of Eq. (23) can be easily found by solving the first order differential equation

$$\left[D - f_1(\phi)\right]\phi = 0. \tag{24}$$

Reyes and Rosu [25] extended this work and found an interesting result that if f_1 is a linear function of ϕ, then Eq. (24) turns out to be Riccati equation and knowing the particular solution ϕ_1 of Eq. (24) facilitate one to obtain its Riccati parameterized general solution $\phi_{\lambda,p}$, which is given as follow

$$\phi_{\lambda,p} = \phi_1 + \frac{e^{I_1}}{\lambda - pI_2}, \tag{25}$$

where

$$I_1(\xi) = \int_{\xi_0}^{\xi}\left(2p\phi_1(\xi') + q\right)d\xi',$$
$$I_2(\xi) = \int_{\xi_0}^{\xi} e^{I_1}(\xi')\,d\xi', \tag{26}$$

and λ is known as Riccati parameter which is to be chosen in such a way so as to avoid singularities. It is also called 'growth parameter' in a sense that it take solution from ϕ_1 to $\phi_{\lambda,p}$.

It can be shown that Eq. (16) can be factorized as Eq. (23), such that

$$f_1(\phi) = p\phi + q \quad \text{and} \quad f_2(\phi) = -(2p\phi + q), \tag{27}$$

where p and q are constants, which satisfies the following relations

$$p^2 = \frac{A}{2} \quad \text{and} \quad q^2 = C, \tag{28}$$

with a constraining condition $B^2 = \frac{9}{2}AC$, which implies $a^2 = \frac{3l_0 - h}{2(l_0 - h)}$. Consider the first order differential equation

$$\left[D - f_1(\phi)\right]\phi = 0, \tag{29}$$

whose solution will also be solution of Eq. (23). Integration of Eq. (29) gives us

$$\phi = -\frac{q}{2p}\left[1 \pm \tanh\left(\frac{q}{2}\xi\right)\right]. \tag{30}$$

Substituting the values of p and q from Eq. (28), we can write one particular solution for Eq. (16) as

$$\phi = -\frac{B}{3A}\left[1 \pm \tanh\left(\frac{\sqrt{C}}{2}\xi\right)\right]. \tag{31}$$

By substituting the values of A, B and C from Eq. (15), we get the same solution for Eq. (16) as we get before by using elliptic function method.

As f_1 comes out to be a linear function of ϕ, we can write down the Riccati parameterized general solution for Eq. (23). Consider the solution (31) with positive sign

$$\phi_1 = -\frac{B}{3A}\left[1 + \tanh\left(\frac{\sqrt{C}}{2}\xi\right)\right], \tag{32}$$

and use Eq. (28), to write I_1 and I_2 from Eq. (26) as

$$I_1 = -2\ln\left[\frac{\cosh(\frac{\sqrt{C}}{2}\xi)}{\cosh(\frac{\sqrt{C}}{2}\xi_0)}\right], \tag{33}$$

$$I_2 = \frac{2}{\sqrt{C}}\cosh^2\left(\frac{\sqrt{C}}{2}\xi_0\right)\left[\tanh\left(\frac{\sqrt{C}}{2}\xi\right) - \tanh\left(\frac{\sqrt{C}}{2}\xi_0\right)\right], \tag{34}$$

and Riccati generalized solution for Eq. (16) as

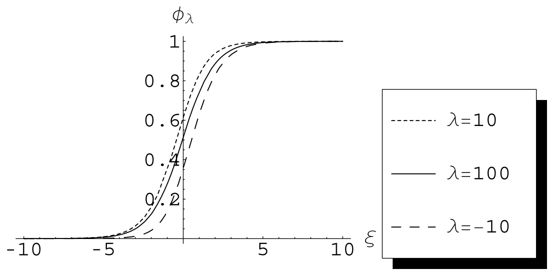

Fig. 1. Graphical representation of ϕ_λ for different values of λ.

$$\phi_\lambda = -\frac{B}{3A}\left[1+\tanh\left(\frac{\sqrt{C}}{2}\xi\right)\right] + \frac{\cosh^2(\frac{\sqrt{C}}{2}\xi_0)\,\mathrm{sech}^2(\frac{\sqrt{C}}{2}\xi)}{\lambda-\frac{3A}{B}\cosh^2(\frac{\sqrt{C}}{2}\xi_0)[\tanh(\frac{\sqrt{C}}{2}\xi)-\tanh(\frac{\sqrt{C}}{2}\xi_0)]}. \quad (35)$$

The graphical representation of typical generalized solution for different values of λ is shown in Fig. 1, which confirms that Eq. (35) is a topological stable solution.

5. Conclusion

In this work, we studied the nonlinear model for DNA by taking into account the longitudinal and transverse motions of DNA. The pair of coupled nonlinear partial differential equations describing the model, decoupled by choosing a linear relation between longitudinal and transverse motions. We found the particular solution and then obtained Riccati generalized solution of solitary kink-type. The Riccati parameter 'λ' plays the role of growth parameter.

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