CHAPTER 15

INTEGRAL TRANSFORMS

15.1 INTEGRAL TRANSFORMS

Frequently in mathematical physics we encounter pairs of functions related by an expression of the form

$$g(\alpha) = \int_a^b f(t) K(\alpha, t)\, dt. \tag{15.1}$$

The function $g(\alpha)$ is called the (integral) transform of $f(t)$ by the kernel $K(\alpha, t)$. The operation may also be described as mapping a function $f(t)$ in t-space into another function, $g(\alpha)$, in α-space. This interpretation takes on physical significance in the time-frequency relation of Fourier transforms, as in Example 15.3.1, and in the real space–momentum space relations in quantum physics of Section 15.6.

Fourier Transform

One of the most useful of the infinite number of possible transforms is the Fourier transform, given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt. \tag{15.2}$$

Two modifications of this form, developed in Section 15.3, are the Fourier cosine and Fourier sine transforms:

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t\, dt, \tag{15.3}$$

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t\, dt. \tag{15.4}$$

The Fourier transform is based on the kernel $e^{i\omega t}$ and its real and imaginary parts taken separately, $\cos\omega t$ and $\sin\omega t$. Because these kernels are the functions used to describe waves, Fourier transforms appear frequently in studies of waves and the extraction of information from waves, particularly when phase information is involved. The output of a stellar interferometer, for instance, involves a Fourier transform of the brightness across a stellar disk. The electron distribution in an atom may be obtained from a Fourier transform of the amplitude of scattered X-rays. In quantum mechanics the physical origin of the Fourier relations of Section 15.6 is the wave nature of matter and our description of matter in terms of waves.

Example 15.1.1 Fourier Transform of Gaussian

The Fourier transform of a Gaussian function $e^{-a^2t^2}$,

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2t^2} e^{i\omega t}\, dt,$$

can be done analytically by completing the square in the exponent,

$$-a^2t^2 + i\omega t = -a^2\left(t - \frac{i\omega}{2a^2}\right)^2 - \frac{\omega^2}{4a^2},$$

which we check by evaluating the square. Substituting this identity we obtain

$$g(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/4a^2} \int_{-\infty}^{\infty} e^{-a^2t^2}\, dt,$$

upon shifting the integration variable $t \to t + \frac{i\omega}{2a^2}$. This is justified by an application of Cauchy's theorem to the rectangle with vertices $-T,\ T,\ T + \frac{i\omega}{2a^2},\ -T + \frac{i\omega}{2a^2}$ for $T \to \infty$, noting that the integrand has no singularities in this region and that the integrals over the sides from $\pm T$ to $\pm T + \frac{i\omega}{2a^2}$ become negligible for $T \to \infty$. Finally we rescale the integration variable as $\xi = at$ in the integral (see Eqs. (8.6) and (8.8)):

$$\int_{-\infty}^{\infty} e^{-a^2t^2}\, dt = \frac{1}{a}\int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi = \frac{\sqrt{\pi}}{a}.$$

Substituting these results we find

$$g(\omega) = \frac{1}{a\sqrt{2}} \exp\left(-\frac{\omega^2}{4a^2}\right),$$

again a Gaussian, but in ω-space. The bigger a is, that is, the narrower the original Gaussian $e^{-a^2t^2}$ is, the wider is its Fourier transform $\sim e^{-\omega^2/4a^2}$. ■

Laplace, Mellin, and Hankel Transforms

Three other useful kernels are

$$e^{-\alpha t}, \qquad t J_n(\alpha t), \qquad t^{\alpha-1}.$$

These give rise to the following transforms

$$g(\alpha) = \int_0^\infty f(t) e^{-\alpha t} dt, \qquad \text{Laplace transform} \tag{15.5}$$

$$g(\alpha) = \int_0^\infty f(t) t J_n(\alpha t)\, dt, \qquad \text{Hankel transform (Fourier–Bessel)} \tag{15.6}$$

$$g(\alpha) = \int_0^\infty f(t) t^{\alpha-1} dt, \qquad \text{Mellin transform.} \tag{15.7}$$

Clearly, the possible types are unlimited. These transforms have been useful in mathematical analysis and in physical applications. We have actually used the Mellin transform without calling it by name; that is, $g(\alpha) = (\alpha - 1)!$ is the Mellin transform of $f(t) = e^{-t}$. See E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 2nd ed., New York: Oxford University Press (1937), for more Mellin transforms. Of course, we could just as well say $g(\alpha) = n!/\alpha^{n+1}$ is the Laplace transform of $f(t) = t^n$. Of the three, the Laplace transform is by far the most used. It is discussed at length in Sections 15.8 to 15.12. The Hankel transform, a Fourier transform for a Bessel function expansion, represents a limiting case of a Fourier–Bessel series. It occurs in potential problems in cylindrical coordinates and has been applied extensively in acoustics.

Linearity

All these integral transforms are linear; that is,

$$\int_a^b \big[c_1 f_1(t) + c_2 f_2(t)\big] K(\alpha, t)\, dt$$

$$= c_1 \int_a^b f_1(t) K(\alpha, t)\, dt + c_2 \int_a^b f_2(t) K(\alpha, t)\, dt, \tag{15.8}$$

$$\int_a^b c f(t) K(\alpha, t)\, dt = c \int_a^b f(t) K(\alpha, t)\, dt, \tag{15.9}$$

where c_1 and c_2 are constants and $f_1(t)$ and $f_2(t)$ are functions for which the transform operation is defined.

Representing our linear integral transform by the operator $\mathcal{L}$, we obtain

$$g(\alpha) = \mathcal{L} f(t). \tag{15.10}$$

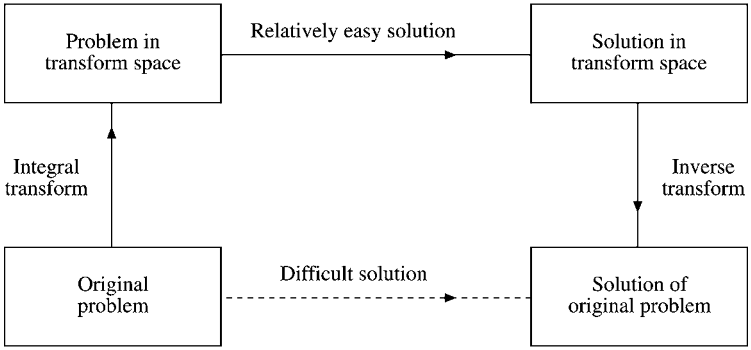

FIGURE 15.1 Schematic integral transforms.

We expect an inverse operator $\mathcal{L}^{-1}$ exists such that[1]

$$f(t) = \mathcal{L}^{-1} g(\alpha). \tag{15.11}$$

For our three Fourier transforms $\mathcal{L}^{-1}$ is given in Section 15.3. In general, the determination of the inverse transform is the main problem in using integral transforms. The inverse Laplace transform is discussed in Section 15.12. For details of the inverse Hankel and inverse Mellin transforms we refer to the Additional Readings at the end of the chapter.

Integral transforms have many special physical applications and interpretations that are noted in the remainder of this chapter. The most common application is outlined in Fig. 15.1. Perhaps an original problem can be solved only with difficulty, if at all, in the original coordinates (space). It often happens that the transform of the problem can be solved relatively easily. Then the inverse transform returns the solution from the transform coordinates to the original system. Example 15.4.1 and Exercise 15.4.1 illustrate this technique.

Exercises

15.1.1 The Fourier transforms for a function of two variables are

$$F(u, v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int f(x, y) e^{i(ux+vy)}\, dx\, dy,$$

$$f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int F(u, v) e^{-i(ux+vy)}\, du\, dv.$$

Using $f(x, y) = f([x^2 + y^2]^{1/2})$, show that the zero-order Hankel transforms

$$F(\rho) = \int_0^{\infty} r f(r) J_0(\rho r)\, dr,$$

$$f(r) = \int_0^{\infty} \rho F(\rho) J_0(\rho r)\, d\rho,$$

are a special case of the Fourier transforms.

[1]Expectation is not proof, and here proof of existence is complicated because we are actually in an **infinite**-dimensional Hilbert space. We shall prove existence in the special cases of interest by actual construction.

This technique may be generalized to derive the Hankel transforms of order $\nu = 0, \frac{1}{2}, 1, \frac{1}{2}, \ldots$ (compare I. N. Sneddon, *Fourier Transforms*, New York: McGraw-Hill (1951)). A more general approach, valid for $\nu > -\frac{1}{2}$, is presented in Sneddon's *The Use of Integral Transforms* (New York: McGraw-Hill (1972)). It might also be noted that the Hankel transforms of nonintegral order $\nu = \pm\frac{1}{2}$ reduce to Fourier sine and cosine transforms.

15.1.2 Assuming the validity of the Hankel transform–inverse transform pair of equations

$$g(\alpha) = \int_0^\infty f(t) J_n(\alpha t) t\, dt,$$

$$f(t) = \int_0^\infty g(\alpha) J_n(\alpha t) \alpha\, d\alpha,$$

show that the Dirac delta function has a Bessel integral representation

$$\delta(t - t') = t \int_0^\infty J_n(\alpha t) J_n(\alpha t') \alpha\, d\alpha.$$

This expression is useful in developing Green's functions in cylindrical coordinates, where the eigenfunctions are Bessel functions.

15.1.3 From the Fourier transforms, Eqs. (15.22) and (15.23), show that the transformation

$$t \to \ln x$$

$$i\omega \to \alpha - \gamma$$

leads to

$$G(\alpha) = \int_0^\infty F(x) x^{\alpha-1} dx$$

and

$$F(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} G(\alpha) x^{-\alpha} d\alpha.$$

These are the Mellin transforms. A similar change of variables is employed in Section 15.12 to derive the inverse Laplace transform.

15.1.4 Verify the following Mellin transforms:

(a) $$\int_0^\infty x^{\alpha-1} \sin(kx)\, dx = k^{-\alpha} (\alpha - 1)! \sin\frac{\pi\alpha}{2}, \qquad -1 < \alpha < 1.$$

(b) $$\int_0^\infty x^{\alpha-1} \cos(kx)\, dx = k^{-\alpha} (\alpha - 1)! \cos\frac{\pi\alpha}{2}, \qquad 0 < \alpha < 1.$$

Hint. You can force the integrals into a tractable form by inserting a convergence factor e^{-bx} and (after integrating) letting $b \to 0$. Also, $\cos kx + i \sin kx = \exp ikx$.

15.2 DEVELOPMENT OF THE FOURIER INTEGRAL

In Chapter 14 it was shown that Fourier series are useful in representing certain functions (1) over a limited range $[0, 2\pi]$, $[-L, L]$, and so on, or (2) for the infinite interval $(-\infty, \infty)$, **if the function is periodic**. We now turn our attention to the problem of representing a nonperiodic function over the infinite range. Physically this means resolving a single pulse or wave packet into sinusoidal waves.

We have seen (Section 14.2) that for the interval $[-L, L]$ the coefficients a_n and b_n could be written as

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt, \tag{15.12}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt. \tag{15.13}$$

The resulting Fourier series is

$$\begin{aligned} f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt &+ \frac{1}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \\ &+ \frac{1}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt, \end{aligned} \tag{15.14}$$

or

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}(t-x)\,dt. \tag{15.15}$$

We now let the parameter L approach infinity, transforming the finite interval $[-L, L]$ into the infinite interval $(-\infty, \infty)$. We set

$$\frac{n\pi}{L} = \omega, \qquad \frac{\pi}{L} = \Delta\omega, \qquad \text{with } L \to \infty.$$

Then we have

$$f(x) \to \frac{1}{\pi}\sum_{n=1}^{\infty}\Delta\omega\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\,dt, \tag{15.16}$$

or

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} d\omega\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\,dt, \tag{15.17}$$

replacing the infinite sum by the integral over ω. The first term (corresponding to a_0) has vanished, assuming that $\int_{-\infty}^{\infty} f(t)\,dt$ exists.

It must be emphasized that this result (Eq. (15.17)) is purely formal. It is not intended as a rigorous derivation, but it can be made rigorous (compare I. N. Sneddon, *Fourier Transforms*, Section 3.2). We take Eq. (15.17) as the Fourier integral. It is subject to the conditions that $f(x)$ is (1) piecewise continuous, (2) piecewise differentiable, and (3) absolutely integrable — that is, $\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite.

Fourier Integral — Exponential Form

Our Fourier integral (Eq. (15.17)) may be put into exponential form by noting that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\,dt, \tag{15.18}$$

whereas

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(t)\sin\omega(t-x)\,dt = 0; \tag{15.19}$$

$\cos\omega(t-x)$ is an even function of ω and $\sin\omega(t-x)$ is an odd function of ω. Adding Eqs. (15.18) and (15.19) (with a factor i), we obtain the **Fourier integral theorem**

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\,d\omega \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt.} \tag{15.20}$$

The variable ω introduced here is an arbitrary mathematical variable. In many physical problems, however, it corresponds to the angular frequency ω. We may then interpret Eq. (15.18) or (15.20) as a representation of $f(x)$ in terms of a distribution of infinitely long sinusoidal wave trains of angular frequency ω, in which this frequency is a **continuous** variable.

Dirac Delta Function Derivation

If the order of integration of Eq. (15.20) is reversed, we may rewrite it as

$$\boxed{f(x) = \int_{-\infty}^{\infty} f(t)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-x)}d\omega\right\}dt.} \tag{15.20a}$$

Apparently the quantity in curly brackets behaves as a delta function $\delta(t-x)$. We might take Eq. (15.20a) as presenting us with a representation of the Dirac delta function. Alternatively, we take it as a clue to a new derivation of the Fourier integral theorem.

From Eq. (1.171b) (shifting the singularity from $t=0$ to $t=x$),

$$f(x) = \lim_{n\to\infty}\int_{-\infty}^{\infty} f(t)\delta_n(t-x)dt, \tag{15.21a}$$

where $\delta_n(t-x)$ is a sequence defining the distribution $\delta(t-x)$. Note that Eq. (15.21a) assumes that $f(t)$ is continuous at $t=x$. We take $\delta_n(t-x)$ to be

$$\delta_n(t-x) = \frac{\sin n(t-x)}{\pi(t-x)} = \frac{1}{2\pi}\int_{-n}^{n} e^{i\omega(t-x)}\,d\omega, \tag{15.21b}$$

using Eq. (1.174). Substituting into Eq. (15.21a), we have

$$f(x) = \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\int_{-n}^{n} e^{i\omega(t-x)}d\omega\,dt. \tag{15.21c}$$

Interchanging the order of integration and then taking the limit as $n \to \infty$, we have Eq. (15.20), the Fourier integral theorem.

With the understanding that it belongs under an integral sign, as in Eq. (15.21a), the identification

$$\boxed{\delta(t-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-x)}\,d\omega} \tag{15.21d}$$

provides a very useful representation of the delta function.

15.3 Fourier Transforms — Inversion Theorem

Let us **define** $g(\omega)$, the Fourier transform of the function $f(t)$, by

$$g(\omega) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt. \tag{15.22}$$

Exponential Transform

Then, from Eq. (15.20), we have the inverse relation,

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{-i\omega t}\,d\omega. \tag{15.23}$$

Note that Eqs. (15.22) and (15.23) are almost but not quite symmetrical, differing in the sign of i.

Here two points deserve comment. First, the $1/\sqrt{2\pi}$ symmetry is a matter of choice, not of necessity. Many authors will attach the entire $1/2\pi$ factor of Eq. (15.20) to one of the two equations: Eq. (15.22) or Eq. (15.23). Second, although the Fourier integral, Eq. (15.20), has received much attention in the mathematics literature, we shall be primarily interested in the Fourier transform and its inverse. They are the equations with physical significance.

When we move the Fourier transform pair to three-dimensional space, it becomes

$$g(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int f(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}\,d^3r, \tag{15.23a}$$

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int g(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}\,d^3k. \tag{15.23b}$$

The integrals are over all space. Verification, if desired, follows immediately by substituting the left-hand side of one equation into the integrand of the other equation and using the three-dimensional delta function.[2] Equation (15.23b) may be interpreted as an expansion of a function $f(\mathbf{r})$ in a continuum of plane wave eigenfunctions; $g(\mathbf{k})$ then becomes the amplitude of the wave, $\exp(-i\mathbf{k}\cdot\mathbf{r})$.

[2] $\delta(\mathbf{r}_1 - \mathbf{r}_2) = \delta(x_1 - x_2)\delta(y_1 - y_2)\delta(z_1 - z_2)$ with Fourier integral $\delta(x_1 - x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[ik_1(x_1 - x_2)]\,dk_1$, etc.

Cosine Transform

If $f(x)$ is odd or even, these transforms may be expressed in a somewhat different form. Consider first an even function f_c with $f_c(x) = f_c(-x)$. Writing the exponential of Eq. (15.22) in trigonometric form, we have

$$\begin{aligned} g_c(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_c(t)(\cos\omega t + i\sin\omega t)\,dt \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(t)\cos\omega t\,dt, \end{aligned} \tag{15.24}$$

the $\sin\omega t$ dependence vanishing on integration over the symmetric interval $(-\infty, \infty)$. Similarly, since $\cos\omega t$ is even, Eqs. (15.23) transforms to

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(\omega)\cos\omega x\,d\omega. \tag{15.25}$$

Equations (15.24) and (15.25) are known as Fourier cosine transforms.

Sine Transform

The corresponding pair of Fourier sine transforms is obtained by assuming that $f_s(x) = -f_s(-x)$, odd, and applying the same symmetry arguments. The equations are

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(t)\sin\omega t\,dt,^{3} \tag{15.26}$$

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(\omega)\sin\omega x\,d\omega. \tag{15.27}$$

From the last equation we may develop the physical interpretation that $f(x)$ is being described by a continuum of sine waves. The amplitude of $\sin\omega x$ is given by $\sqrt{2/\pi}\,g_s(\omega)$, in which $g_s(\omega)$ is the Fourier sine transform of $f(x)$. It will be seen that Eq. (15.27) is the integral analog of the summation (Eq. (14.24)). Similar interpretations hold for the cosine and exponential cases.

If we take Eqs. (15.22), (15.24), and (15.26) as the direct integral transforms, described by $\mathcal{L}$ in Eq. (15.10) (Section 15.1), the corresponding inverse transforms, $\mathcal{L}^{-1}$ of Eq. (15.11), are given by Eqs. (15.23), (15.25), and (15.27).

Note that the Fourier cosine transforms and the Fourier sine transforms each involve only positive values (and zero) of the arguments. We use the parity of $f(x)$ to establish the transforms; but once the transforms are established, the behavior of the functions f and g for negative argument is irrelevant. In effect, the transform equations themselves impose a **definite parity**: **even for the Fourier cosine** transform and **odd for the Fourier sine** transform.

[3]Note that a factor $-i$ has been absorbed into this $g(\omega)$.

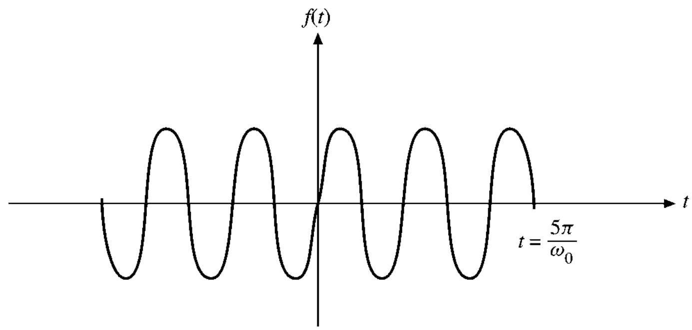

FIGURE 15.2 Finite wave train.

Example 15.3.1 FINITE WAVE TRAIN

An important application of the Fourier transform is the resolution of a finite pulse into sinusoidal waves. Imagine that an infinite wave train $\sin\omega_0 t$ is clipped by Kerr cell or saturable dye cell shutters so that we have

$$f(t) = \begin{cases} \sin\omega_0 t, & |t| < \dfrac{N\pi}{\omega_0}, \\ 0, & |t| > \dfrac{N\pi}{\omega_0}. \end{cases} \tag{15.28}$$

This corresponds to N cycles of our original wave train (Fig. 15.2). Since $f(t)$ is odd, we may use the Fourier sine transform (Eq. (15.26)) to obtain

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{N\pi/\omega_0} \sin\omega_0 t \sin\omega t \, dt. \tag{15.29}$$

Integrating, we find our amplitude function:

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \left[\frac{\sin[(\omega_0 - \omega)(N\pi/\omega_0)]}{2(\omega_0 - \omega)} - \frac{\sin[(\omega_0 + \omega)(N\pi/\omega_0)]}{2(\omega_0 + \omega)} \right]. \tag{15.30}$$

It is of considerable interest to see how $g_s(\omega)$ depends on frequency. For large ω_0 and $\omega \approx \omega_0$, only the first term will be of any importance because of the denominators. It is plotted in Fig. 15.3. This is the amplitude curve for the single-slit diffraction pattern.

There are zeros at

$$\frac{\omega_0 - \omega}{\omega_0} = \frac{\Delta\omega}{\omega_0} = \pm\frac{1}{N}, \pm\frac{2}{N}, \qquad \text{and so on.} \tag{15.31}$$

For large N, $g_s(\omega)$ may also be interpreted as a Dirac delta distribution, as in Section 1.15. Since the contributions outside the central maximum are small in this case, we may take

$$\Delta\omega = \frac{\omega_0}{N} \tag{15.32}$$

as a good measure of the spread in frequency of our wave pulse. Clearly, if N is large (a long pulse), the frequency spread will be small. On the other hand, if our pulse is clipped

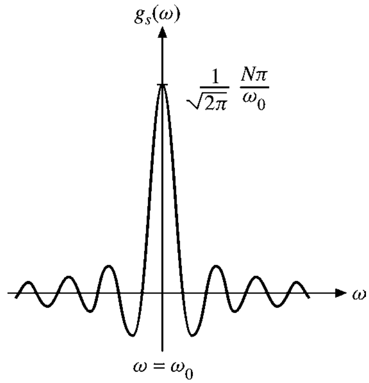

FIGURE 15.3 Fourier transform of finite wave train.

short, N small, the frequency distribution will be wider and the secondary maxima are more important. ■

Uncertainty Principle

Here is a classical analog of the famous uncertainty principle of quantum mechanics. If we are dealing with electromagnetic waves,

$$\frac{h\omega}{2\pi} = E, \qquad \text{energy (of our photon)}$$

$$\frac{h\Delta\omega}{2\pi} = \Delta E, \tag{15.33}$$

h being Planck's constant. Here ΔE represents an uncertainty in the energy of our pulse. There is also an uncertainty in the time, for our wave of N cycles requires $2N\pi/\omega_0$ seconds to pass. Taking

$$\Delta t = \frac{2N\pi}{\omega_0}, \tag{15.34}$$

we have the product of these two uncertainties:

$$\Delta E \cdot \Delta t = \frac{h\,\Delta\omega}{2\pi} \cdot \frac{2\pi N}{\omega_0} = h\frac{\omega_0}{2\pi N} \cdot \frac{2\pi N}{\omega_0} = h. \tag{15.35}$$

The Heisenberg uncertainty principle actually states

$$\Delta E \cdot \Delta t \geq \frac{h}{4\pi}, \tag{15.36}$$

and this is clearly satisfied in our example.

Exercises

15.3.1 (a) Show that $g(-\omega) = g^*(\omega)$ is a necessary and sufficient condition for $f(x)$ to be real.

(b) Show that $g(-\omega) = -g^*(\omega)$ is a necessary and sufficient condition for $f(x)$ to be pure imaginary.

Note. The condition of part (a) is used in the development of the dispersion relations of Section 7.2.

15.3.2 Let $F(\omega)$ be the Fourier (exponential) transform of $f(x)$ and $G(\omega)$ be the Fourier transform of $g(x) = f(x + a)$. Show that

$$G(\omega) = e^{-ia\omega} F(\omega).$$

15.3.3 The function

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

is a symmetrical finite step function.

(a) Find the $g_c(\omega)$, Fourier cosine transform of $f(x)$.

(b) Taking the inverse cosine transform, show that

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin\omega \cos\omega x}{\omega} d\omega.$$

(c) From part (b) show that

$$\int_0^\infty \frac{\sin\omega \cos\omega x}{\omega} d\omega = \begin{cases} 0, & |x| > 1, \\ \frac{\pi}{4}, & |x| = 1, \\ \frac{\pi}{2}, & |x| < 1. \end{cases}$$

15.3.4 (a) Show that the Fourier sine and cosine transforms of e^{-at} are

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \frac{\omega}{\omega^2 + a^2}, \qquad g_c(\omega) = \sqrt{\frac{2}{\pi}} \frac{a}{\omega^2 + a^2}.$$

Hint. Each of the transforms can be related to the other by integration by parts.

(b) Show that

$$\int_0^\infty \frac{\omega \sin\omega x}{\omega^2 + a^2} d\omega = \frac{\pi}{2} e^{-ax}, \qquad x > 0,$$

$$\int_0^\infty \frac{\cos\omega x}{\omega^2 + a^2} d\omega = \frac{\pi}{2a} e^{-ax}, \qquad x > 0.$$

These results are also obtained by contour integration (Exercise 7.1.14).

15.3.5 Find the Fourier transform of the triangular pulse (Fig. 15.4).

$$f(x) = \begin{cases} h\bigl(1 - a|x|\bigr), & |x| < \frac{1}{a}, \\ 0, & |x| > \frac{1}{a}. \end{cases}$$

Note. This function provides another delta sequence with $h = a$ and $a \to \infty$.

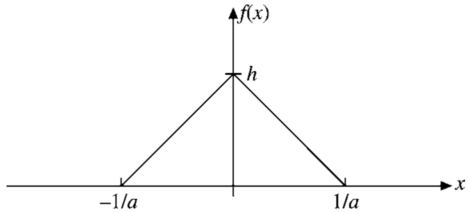

FIGURE 15.4 Triangular pulse.

15.3.6 Define a sequence

$$\delta_n(x) = \begin{cases} n, & |x| < \frac{1}{2n}, \\ 0, & |x| > \frac{1}{2n}. \end{cases}$$

(This is Eq. (1.172).) Express $\delta_n(x)$ as a Fourier integral (via the Fourier integral theorem, inverse transform, etc.). Finally, show that we may write

$$\delta(x) = \lim_{n\to\infty} \delta_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\, dk.$$

15.3.7 Using the sequence

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} \exp\left(-n^2x^2\right),$$

show that

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\, dk.$$

Note. Remember that $\delta(x)$ is defined in terms of its behavior as part of an integrand (Section 1.15), especially Eqs. (1.178) and (1.179).

15.3.8 Derive sine and cosine representations of $\delta(t-x)$ that are comparable to the exponential representation, Eq. (15.21d).

ANS. $\dfrac{2}{\pi}\displaystyle\int_0^{\infty} \sin\omega t \sin\omega x\, d\omega, \quad \dfrac{2}{\pi}\displaystyle\int_0^{\infty} \cos\omega t \cos\omega x\, d\omega.$

15.3.9 In a resonant cavity an electromagnetic oscillation of frequency ω_0 dies out as

$$A(t) = A_0 e^{-\omega_0 t/2Q} e^{-i\omega_0 t}, \qquad t > 0.$$

(Take $A(t) = 0$ for $t < 0$.) The parameter Q is a measure of the ratio of stored energy to energy loss per cycle. Calculate the frequency distribution of the oscillation, $a^*(\omega)a(\omega)$, where $a(\omega)$ is the Fourier transform of $A(t)$.
Note. The larger Q is, the sharper your resonance line will be.

ANS. $a^*(\omega)a(\omega) = \dfrac{A_0^2}{2\pi}\dfrac{1}{(\omega-\omega_0)^2 + (\omega_0/2Q)^2}.$

15.3.10 Prove that

$$\frac{\hbar}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}\,d\omega}{E_0 - i\Gamma/2 - \hbar\omega} = \begin{cases} \exp\left(-\frac{\Gamma t}{2\hbar}\right)\exp\left(-\frac{iE_0 t}{\hbar}\right), & t > 0, \\ 0, & t < 0. \end{cases}$$

This Fourier integral appears in a variety of problems in quantum mechanics: WKB barrier penetration, scattering, time-dependent perturbation theory, and so on.
Hint. Try contour integration.

15.3.11 Verify that the following are Fourier integral transforms of one another:

(a) $\sqrt{\frac{2}{\pi}}\cdot\frac{1}{\sqrt{a^2 - x^2}}, \quad |x| < a,$ and $J_0(ay)$,
$0, \quad |x| > a,$

(b) $0, \quad |x| < a,$
$-\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{x^2 + a^2}}, \quad |x| > a,$ and $N_0(a|y|)$,

(c) $\sqrt{\frac{\pi}{2}}\cdot\frac{1}{\sqrt{x^2 + a^2}}$ and $K_0(a|y|)$.

(d) Can you suggest why $I_0(ay)$ is not included in this list?

Hint. J_0, N_0, and K_0 may be transformed most easily by using an exponential representation, reversing the order of integration, and employing the Dirac delta function exponential representation (Section 15.2). These cases can be treated equally well as Fourier cosine transforms.
Note. The K_0 relation appears as a consequence of a Green's function equation in Exercise 9.7.14.

15.3.12 A calculation of the magnetic field of a circular current loop in circular cylindrical coordinates leads to the integral

$$\int_0^{\infty}\cos kz\, k\, K_1(ka)\,dk.$$

Show that this integral is equal to

$$\frac{\pi a}{2(z^2 + a^2)^{3/2}}.$$

Hint. Try differentiating Exercise 15.3.11(c).

15.3.13 As an extension of Exercise 15.3.11, show that

(a) $\displaystyle\int_0^{\infty} J_0(y)\,dy = 1,$ (b) $\displaystyle\int_0^{\infty} N_0(y)\,dy = 0,$ (c) $\displaystyle\int_0^{\infty} K_0(y)\,dy = \frac{\pi}{2}.$

15.3.14 The Fourier integral, Eq. (15.18), has been held meaningless for $f(t) = \cos\alpha t$. Show that the Fourier integral can be extended to cover $f(t) = \cos\alpha t$ by use of the Dirac delta function.

15.3.15 Show that

$$\int_0^\infty \sin ka\, J_0(k\rho)\, dk = \begin{cases} \left(a^2-\rho^2\right)^{-1/2}, & \rho < a, \\ 0, & \rho > a. \end{cases}$$

Here a and ρ are positive. The equation comes from the determination of the distribution of charge on an isolated conducting disk, radius a. Note that the function on the right has an **infinite** discontinuity at $\rho = a$.
Note. A Laplace transform approach appears in Exercise 15.10.8.

15.3.16 The function $f(r)$ has a Fourier exponential transform,

$$g(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3r = \frac{1}{(2\pi)^{3/2} k^2}.$$

Determine $f(\mathbf{r})$.
Hint. Use spherical polar coordinates in k-space.

ANS. $f(\mathbf{r}) = \dfrac{1}{4\pi r}$.

15.3.17 (a) Calculate the Fourier exponential transform of $f(x) = e^{-a|x|}$.
(b) Calculate the inverse transform by employing the calculus of residues (Section 7.1).

15.3.18 Show that the following are Fourier transforms of each other

$$i^n J_n(t) \qquad \text{and} \qquad \begin{cases} \sqrt{\dfrac{2}{\pi}}\, T_n(x)\left(1-x^2\right)^{-1/2}, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

$T_n(x)$ is the nth-order Chebyshev polynomial.
Hint. With $T_n(\cos\theta) = \cos n\theta$, the transform of $T_n(x)(1-x^2)^{-1/2}$ leads to an integral representation of $J_n(t)$.

15.3.19 Show that the Fourier exponential transform of

$$f(\mu) = \begin{cases} P_n(\mu), & |\mu| \le 1, \\ 0, & |\mu| > 1 \end{cases}$$

is $(2i^n/2\pi) j_n(kr)$. Here $P_n(\mu)$ is a Legendre polynomial and $j_n(kr)$ is a spherical Bessel function.

15.3.20 Show that the three-dimensional Fourier exponential transform of a radially symmetric function may be rewritten as a Fourier sine transform:

$$\frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} f(r) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3x = \frac{1}{k}\sqrt{\frac{2}{\pi}} \int_0^\infty \left[r f(r)\right] \sin kr\, dr.$$

15.3.21 (a) Show that $f(x) = x^{-1/2}$ is a **self-reciprocal** under both Fourier cosine and sine transforms; that is,

$$\sqrt{\frac{2}{\pi}} \int_0^\infty x^{-1/2} \cos xt\, dx = t^{-1/2},$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty x^{-1/2} \sin xt\, ds = t^{-1/2}.$$

(b) Use the preceding results to evaluate the Fresnel integrals $\int_0^\infty \cos(y^2)\, dy$ and $\int_0^\infty \sin(y^2)\, dy$.

15.4 Fourier Transform of Derivatives

In Section 15.1, Fig. 15.1 outlines the overall technique of using Fourier transforms and inverse transforms to solve a problem. Here we take an initial step in solving a differential equation — obtaining the Fourier transform of a derivative.

Using the exponential form, we determine that the Fourier transform of $f(x)$ is

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx \tag{15.37}$$

and for $df(x)/dx$

$$g_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df(x)}{dx} e^{i\omega x}\, dx. \tag{15.38}$$

Integrating Eq. (15.38) by parts, we obtain

$$g_1(\omega) = \frac{e^{i\omega x}}{\sqrt{2\pi}} f(x)\Big|_{-\infty}^{\infty} - \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx. \tag{15.39}$$

If $f(x)$ vanishes[4] as $x \to \pm\infty$, we have

$$g_1(\omega) = -i\omega\, g(\omega); \tag{15.40}$$

that is, the transform of the derivative is $(-i\omega)$ times the transform of the original function. This may readily be generalized to the nth derivative to yield

$$g_n(\omega) = (-i\omega)^n g(\omega), \tag{15.41}$$

provided all the integrated parts vanish as $x \to \pm\infty$. This is the power of the Fourier transform, the reason it is so useful in solving (partial) differential equations. The operation of differentiation has been replaced by a multiplication in ω-space.

[4] Apart from cases such as Exercise 15.3.6, $f(x)$ must vanish as $x \to \pm\infty$ in order for the Fourier transform of $f(x)$ to exist.

Example 15.4.1 WAVE EQUATION

This technique may be used to advantage in handling PDEs. To illustrate the technique, let us derive a familiar expression of elementary physics. An infinitely long string is vibrating freely. The amplitude y of the (small) vibrations satisfies the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}. \tag{15.42}$$

We shall assume an initial condition

$$y(x,0) = f(x), \tag{15.43}$$

where f is localized, that is, approaches zero at large x.

Applying our Fourier transform in x, which means multiplying by $e^{i\alpha x}$ and integrating over x, we obtain

$$\int_{-\infty}^{\infty} \frac{\partial^2 y(x,t)}{\partial x^2} e^{i\alpha x}\,dx = \frac{1}{v^2}\int_{-\infty}^{\infty} \frac{\partial^2 y(x,t)}{\partial t^2} e^{i\alpha x}\,dx \tag{15.44}$$

or

$$(-i\alpha)^2 Y(\alpha,t) = \frac{1}{v^2}\frac{\partial^2 Y(\alpha,t)}{\partial t^2}. \tag{15.45}$$

Here we have used

$$Y(\alpha,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(x,t)e^{i\alpha x}\,dx \tag{15.46}$$

and Eq. (15.41) for the second derivative. Note that the integrated part of Eq. (15.39) vanishes: The wave has not yet gone to $\pm\infty$ because it is propagating forward in time, and there is no source at infinity because $f(\pm\infty) = 0$. Since no derivatives with respect to α appear, Eq. (15.45) is actually an ODE—in fact, the linear oscillator equation. This transformation, from a PDE to an ODE, is a significant achievement. We solve Eq. (15.45) subject to the appropriate initial conditions. At $t = 0$, applying Eq. (15.43), Eq. (15.46) reduces to

$$Y(\alpha,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx = F(\alpha). \tag{15.47}$$

The general solution of Eq. (15.45) in exponential form is

$$Y(\alpha,t) = F(\alpha)e^{\pm i v\alpha t}. \tag{15.48}$$

Using the inversion formula (Eq. (15.23)), we have

$$y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Y(\alpha,t)e^{-i\alpha x}\,d\alpha, \tag{15.49}$$

and, by Eq. (15.48),

$$y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha(x\mp vt)}\,d\alpha. \tag{15.50}$$

Since $f(x)$ is the Fourier inverse transform of $F(\alpha)$,

$$y(x,t) = f(x \mp vt), \tag{15.51}$$

corresponding to waves advancing in the $+x$- and $-x$-directions, respectively.

The particular linear combinations of waves is given by the boundary condition of Eq. (15.43) and some other boundary condition, such as a restriction on $\partial y/\partial t$. ■

The accomplishment of the Fourier transform here deserves special emphasis.

- Our Fourier transform converted a PDE into an ODE, where the "degree of transcendence" of the problem was reduced.

In Section 15.9 Laplace transforms are used to convert ODEs (with constant coefficients) into algebraic equations. Again, the degree of transcendence is reduced. The problem is simplified — as outlined in Fig. 15.1.

Example 15.4.2 Heat Flow PDE

To illustrate another transformation of a PDE into an ODE, let us Fourier transform the heat flow partial differential equation

$$\frac{\partial \psi}{\partial t} = a^2 \frac{\partial^2 \psi}{\partial x^2},$$

where the solution $\psi(x,t)$ is the temperature in space as a function of time. By taking the Fourier transform of both sides of this equation (note that here only ω is the transform variable conjugate to x because t is the time in the heat flow PDE), where

$$\Psi(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t) e^{i\omega x}\, dx,$$

this yields an ODE for the Fourier transform Ψ of ψ in the time variable t,

$$\frac{\partial \Psi(\omega,t)}{\partial t} = -a^2\omega^2 \Psi(\omega,t).$$

Integrating we obtain

$$\ln \Psi = -a^2\omega^2 t + \ln C, \qquad \text{or} \qquad \Psi = C e^{-a^2\omega^2 t},$$

where the integration constant C may still depend on ω and, in general, is determined by initial conditions. In fact, $C = \Psi(\omega, 0)$ is the initial spatial distribution of Ψ, so it is given by the transform (in x) of the initial distribution of ψ, namely, $\psi(x,0)$. Putting this solution back into our inverse Fourier transform, this yields

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(\omega) e^{-i\omega x} e^{-a^2\omega^2 t}\, d\omega.$$

For simplicity, we here take C ω-independent (assuming a delta-function initial temperature distribution) and integrate by completing the square in ω, as in Example 15.1.1, making appropriate changes of variables and parameters ($a^2 \to a^2 t,\ \omega \to x, t \to -\omega$). This yields the particular solution of the heat flow PDE,

$$\psi(x,t) = \frac{C}{a\sqrt{2t}} \exp\left(-\frac{x^2}{4a^2 t}\right),$$

which appears as a clever guess in Chapter 8. In effect, we have shown that ψ is the inverse Fourier transform of $C\exp(-a^2\omega^2 t)$. ■

Example 15.4.3 Inversion of PDE

Derive a Fourier integral for the Green's function G_0 of Poisson's PDE, which is a solution of

$$\nabla^2 G_0(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}').$$

Once G_0 is known, the general solution of Poisson's PDE,

$$\nabla^2 \Phi = -4\pi\rho(\mathbf{r})$$

of electrostatics, is given as

$$\Phi(\mathbf{r}) = \int G_0(\mathbf{r}, \mathbf{r}')4\pi\rho(\mathbf{r}')\, d^3r'.$$

Applying ∇^2 to Φ and using the PDE the Green's function satisfies, we check that

$$\nabla^2\Phi(\mathbf{r}) = \int \nabla^2 G_0(\mathbf{r}, \mathbf{r}')4\pi\rho(\mathbf{r}')\, d^3r' = -\int \delta(\mathbf{r} - \mathbf{r}')4\pi\rho(\mathbf{r}')\, d^3r' = -4\pi\rho(\mathbf{r}).$$

Now we use the Fourier transform of G_0, which is g_0, and of that of the δ function, writing

$$\nabla^2 \int g_0(\mathbf{p}) e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} \frac{d^3p}{(2\pi)^3} = -\int e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} \frac{d^3p}{(2\pi)^3}.$$

Because the integrands of equal Fourier integrals must be the same (almost) everywhere, which follows from the inverse Fourier transform, and with

$$\nabla e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} = i\mathbf{p} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')},$$

this yields $-\mathbf{p}^2 g_0(\mathbf{p}) = -1$. Therefore, application of the Laplacian to a Fourier integral $f(\mathbf{r})$ corresponds to multiplying its Fourier transform $g(\mathbf{p})$ by $-\mathbf{p}^2$. Substituting this solution into the inverse Fourier transform for G_0 gives

$$G_0(\mathbf{r}, \mathbf{r}') = \int e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} \frac{d^3p}{(2\pi)^3 \mathbf{p}^2} = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}.$$

We can verify the last part of this result by applying ∇^2 to G_0 again and recalling from Chapter 1 that $\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r} - \mathbf{r}')$.

The inverse Fourier transform can be evaluated using polar coordinates, exploiting the spherical symmetry of $\mathbf{p}^2$. For simplicity, we write $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and call θ the angle between $\mathbf{R}$ and $\mathbf{p}$,

$$\begin{aligned}\int e^{i\mathbf{p}\cdot\mathbf{R}}\frac{d^3p}{p^2} &= \int_0^\infty dp \int_{-1}^1 e^{ipR\cos\theta} d\cos\theta \int_0^{2\pi} d\varphi \\ &= \frac{2\pi}{iR}\int_0^\infty \frac{dp}{p} e^{ipR\cos\theta}\Big|_{\cos\theta=-1}^1 = \frac{4\pi}{R}\int_0^\infty \frac{\sin pR}{p}dp \\ &= \frac{4\pi}{R}\int_0^\infty \frac{\sin pR}{pR} d(pR) = \frac{2\pi^2}{R},\end{aligned}$$

where θ and φ are the angles of $\mathbf{p}$ and $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$, from Example 7.1.4. Dividing by $(2\pi)^3$, we obtain $G_0(R) = 1/(4\pi R)$, as claimed. An evaluation of this Fourier transform by contour integration is given in Example 9.7.2. ■

Exercises

15.4.1 The one-dimensional Fermi age equation for the diffusion of neutrons slowing down in some medium (such as graphite) is

$$\frac{\partial^2 q(x,\tau)}{\partial x^2} = \frac{\partial q(x,\tau)}{\partial \tau}.$$

Here q is the number of neutrons that slow down, falling below some given energy per second per unit volume. The Fermi age, τ, is a measure of the energy loss.
If $q(x,0) = S\delta(x)$, corresponding to a plane source of neutrons at $x = 0$, emitting S neutrons per unit area per second, derive the solution

$$q = S\frac{e^{-x^2/4\tau}}{\sqrt{4\pi\tau}}.$$

Hint. Replace $q(x,\tau)$ with

$$p(k,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty q(x,\tau)e^{ikx}\,dx.$$

This is analogous to the diffusion of heat in an infinite medium.

15.4.2 Equation (15.41) yields

$$g_2(\omega) = -\omega^2 g(\omega)$$

for the Fourier transform of the second derivative of $f(x)$. The condition $f(x) \to 0$ for $x \to \pm\infty$ may be relaxed slightly. Find the least restrictive condition for the preceding equation for $g_2(\omega)$ to hold.

$$\text{ANS. } \left[\frac{df(x)}{dx} - i\omega f(x)\right]e^{i\omega x}\Big|_{-\infty}^{\infty} = 0.$$

15.4.3 The one-dimensional neutron diffusion equation with a (plane) source is

$$-D\frac{d^2\varphi(x)}{dx^2} + K^2 D\varphi(x) = Q\delta(x),$$

where $\varphi(x)$ is the neutron flux, $Q\delta(x)$ is the (plane) source at $x = 0$, and D and K^2 are constants. Apply a Fourier transform. Solve the equation in transform space. Transform your solution back into x-space.

ANS. $\varphi(x) = \dfrac{Q}{2KD}e^{-|Kx|}$.

15.4.4 For a point source at the origin, the three-dimensional neutron diffusion equation becomes

$$-D\nabla^2\varphi(\mathbf{r}) + K^2 D\varphi(\mathbf{r}) = Q\delta(\mathbf{r}).$$

Apply a three-dimensional Fourier transform. Solve the transformed equation. Transform the solution back into **r**-space.

15.4.5 (a) Given that $F(\mathbf{k})$ is the three-dimensional Fourier transform of $f(\mathbf{r})$ and $F_1(\mathbf{k})$ is the three-dimensional Fourier transform of $\nabla f(\mathbf{r})$, show that

$$F_1(\mathbf{k}) = (-i\mathbf{k})F(\mathbf{k}).$$

This is a three-dimensional generalization of Eq. (15.40).

(b) Show that the three-dimensional Fourier transform of $\nabla \cdot \nabla f(\mathbf{r})$ is

$$F_2(\mathbf{k}) = (-i\mathbf{k})^2 F(\mathbf{k}).$$

Note. Vector **k** is a vector in the transform space. In Section 15.6 we shall have $\hbar\mathbf{k} = \mathbf{p}$, linear momentum.

15.5 Convolution Theorem

We shall employ convolutions to solve differential equations, to normalize momentum wave functions (Section 15.6), and to investigate transfer functions (Section 15.7).

Let us consider two functions $f(x)$ and $g(x)$ with Fourier transforms $F(t)$ and $G(t)$, respectively. We define the operation

$$f * g \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)f(x-y)\,dy \tag{15.52}$$

as the **convolution** of the two functions f and g over the interval $(-\infty, \infty)$. This form of an integral appears in probability theory in the determination of the probability density of two random, independent variables. Our solution of Poisson's equation, Eq. (9.148), may be interpreted as a convolution of a charge distribution, $\rho(\mathbf{r}_2)$, and a weighting function, $(4\pi\varepsilon_0|\mathbf{r}_1 - \mathbf{r}_2|)^{-1}$. In other works this is sometimes referred to as the **Faltung**, to use the

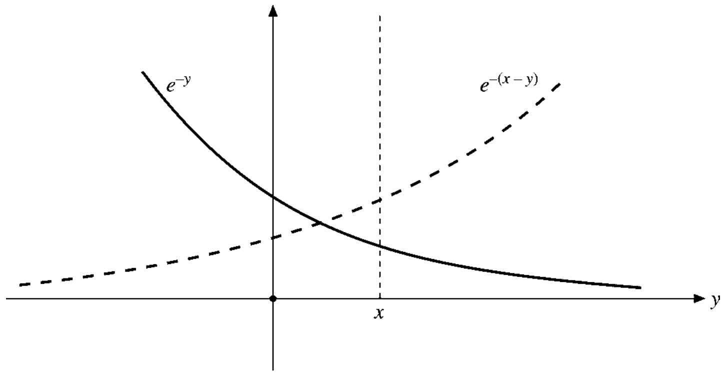

FIGURE 15.5

German term for "folding."[5] We now transform the integral in Eq. (15.52) by introducing the Fourier transforms:

$$\begin{aligned}\int_{-\infty}^{\infty} g(y)f(x-y)dy &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} F(t)e^{-it(x-y)}\,dt\,dy\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(t)\left[\int_{-\infty}^{\infty} g(y)e^{ity}dy\right]e^{-itx}\,dt\\ &= \int_{-\infty}^{\infty} F(t)G(t)e^{-itx}\,dt, \end{aligned} \tag{15.53}$$

interchanging the order of integration and transforming $g(y)$. This result may be interpreted as follows: The Fourier inverse transform of a **product** of Fourier transforms is the convolution of the original functions, $f * g$.

For the special case $x = 0$ we have

$$\int_{-\infty}^{\infty} F(t)G(t)\,dt = \int_{-\infty}^{\infty} f(-y)g(y)\,dy. \tag{15.54}$$

The minus sign in $-y$ suggests that modifications be tried. We now do this with g^* instead of g using a different technique.

Parseval's Relation

Results analogous to Eqs. (15.53) and (15.54) may be derived for the Fourier sine and cosine transforms (Exercises 15.5.1 and 15.5.3). Equation (15.54) and the corresponding sine and cosine convolutions are often labeled *Parseval's relations* by analogy with Parseval's theorem for Fourier series (Chapter 14, Exercise 14.4.2).

[5]For $f(y) = e^{-y}$, $f(y)$ and $f(x-y)$ are plotted in Fig. 15.5. Clearly, $f(y)$ and $f(x-y)$ are mirror images of each other in relation to the vertical line $y = x/2$, that is, we could generate $f(x-y)$ by folding over $f(y)$ on the line $y = x/2$.

The Parseval relation[6,7]

$$\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\,d\omega = \int_{-\infty}^{\infty} f(t)g^*(t)\,dt \tag{15.55}$$

may be derived elegantly using the Dirac delta function representation, Eq. (15.21d). We have

$$\int_{-\infty}^{\infty} f(t)g^*(t)\,dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G^*(x)e^{ixt}\,dx\,dt, \tag{15.56}$$

with attention to the complex conjugation in the $G^*(x)$ to $g^*(t)$ transform. Integrating over t first, and using Eq. (15.21d), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)g^*(t)\,dt &= \int_{-\infty}^{\infty} F(\omega)\int_{-\infty}^{\infty} G^*(x)\delta(x-\omega)\,dx\,d\omega \\ &= \int_{-\infty}^{\infty} F(\omega)G^*(\omega)\,d\omega,\end{aligned} \tag{15.57}$$

our desired Parseval relation. If $f(t) = g(t)$, then the integrals in the Parseval relation are normalization integrals (Section 10.4). Equation (15.57) guarantees that if a function $f(t)$ is normalized to unity, its transform $F(\omega)$ is likewise normalized to unity. This is extremely important in quantum mechanics as developed in the next section.

It may be shown that the Fourier transform is a unitary operation (in the Hilbert space L^2, square integrable functions). The Parseval relation is a reflection of this unitary property — analogous to Exercise 3.4.26 for matrices.

In Fraunhofer diffraction optics the diffraction pattern (amplitude) appears as the transform of the function describing the aperture (compare Exercise 15.5.5). With intensity proportional to the square of the amplitude the Parseval relation implies that the energy passing through the aperture seems to be somewhere in the diffraction pattern — a statement of the conservation of energy. Parseval's relations may be developed independently of the inverse Fourier transform and then used rigorously to derive the inverse transform. Details are given by Morse and Feshbach,[8] Section 4.8 (see also Exercise 15.5.4).

Exercises

15.5.1 Work out the convolution equation corresponding to Eq. (15.53) for

(a) Fourier sine transforms

$$\frac{1}{2}\int_0^{\infty} g(y)\big[f(y+x) + f(y-x)\big]\,dy = \int_0^{\infty} F_s(s)G_s(s)\cos sx\,ds,$$

where f and g are odd functions.

[6] Note that all arguments are positive, in contrast to Eq. (15.54).

[7] Some authors prefer to restrict Parseval's name to series and refer to Eq. (15.55) as **Rayleigh's theorem**.

[8] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, New York: McGraw-Hill (1953).

(b) Fourier cosine transforms

$$\frac{1}{2}\int_0^\infty g(y)\big[f(y+x)+f(x-y)\big]\,dy=\int_0^\infty F_c(s)G_c(s)\cos sx\,ds,$$

where f and g are even functions.

15.5.2 $F(\rho)$ and $G(\rho)$ are the Hankel transforms of $f(r)$ and $g(r)$, respectively (Exercise 15.1.1). Derive the Hankel transform Parseval relation:

$$\int_0^\infty F^*(\rho)G(\rho)\rho\,d\rho=\int_0^\infty f^*(r)g(r)r\,dr.$$

15.5.3 Show that for both Fourier sine and Fourier cosine transforms Parseval's relation has the form

$$\int_0^\infty F(t)G(t)\,dt=\int_0^\infty f(y)g(y)\,dy.$$

15.5.4 Starting from Parseval's relation (Eq. (15.54)), let $g(y)=1$, $0\le y\le\alpha$, and zero elsewhere. From this derive the Fourier inverse transform (Eq. (15.23)).
Hint. Differentiate with respect to α.

15.5.5 (a) A rectangular pulse is described by

$$f(x)=\begin{cases}1, & |x|<a,\\ 0, & |x|>a.\end{cases}$$

Show that the Fourier exponential transform is

$$F(t)=\sqrt{\frac{2}{\pi}}\frac{\sin at}{t}.$$

This is the single-slit diffraction problem of physical optics. The slit is described by $f(x)$. The diffraction pattern **amplitude** is given by the Fourier transform $F(t)$.

(b) Use the Parseval relation to evaluate

$$\int_{-\infty}^\infty \frac{\sin^2 t}{t^2}\,dt.$$

This integral may also be evaluated by using the calculus of residues, Exercise 7.1.12.

ANS. (b) π.

15.5.6 Solve Poisson's equation, $\nabla^2\psi(\mathbf{r})=-\rho(\mathbf{r})/\varepsilon_0$, by the following sequence of operations:

(a) Take the Fourier transform of both sides of this equation. Solve for the Fourier transform of $\psi(\mathbf{r})$.

(b) Carry out the Fourier inverse transform by using a three-dimensional analog of the convolution theorem, Eq. (15.53).

15.5.7 (a) Given $f(x) = 1 - |x/2|$, $-2 \le x \le 2$, and zero elsewhere, show that the Fourier transform of $f(x)$ is

$$F(t) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin t}{t}\right)^2.$$

(b) Using the Parseval relation, evaluate

$$\int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^4 dt.$$

ANS. (b) $\dfrac{2\pi}{3}$.

15.5.8 With $F(t)$ and $G(t)$ the Fourier transforms of $f(x)$ and $g(x)$, respectively, show that

$$\int_{-\infty}^{\infty} \left|f(x) - g(x)\right|^2 dx = \int_{-\infty}^{\infty} \left|F(t) - G(t)\right|^2 dt.$$

If $g(x)$ is an approximation to $f(x)$, the preceding relation indicates that the mean square deviation in t-space is equal to the mean square deviation in x-space.

15.5.9 Use the Parseval relation to evaluate

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{d\omega}{(\omega^2 + a^2)^2}$, (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\omega^2 d\omega}{(\omega^2 + a^2)^2}$.

Hint. Compare Exercise 15.3.4.

ANS. (a) $\dfrac{\pi}{2a^3}$, (b) $\dfrac{\pi}{2a}$.

15.6 MOMENTUM REPRESENTATION

In advanced dynamics and in quantum mechanics, linear momentum and spatial position occur on an equal footing. In this section we shall start with the usual space distribution and derive the corresponding momentum distribution. For the one-dimensional case our wave function $\psi(x)$ has the following properties:

1. $\psi^*(x)\psi(x)\,dx$ is the probability density of finding a quantum particle between x and $x + dx$, and
2. $$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,dx = 1 \qquad (15.58)$$
   corresponds to probability unity.
3. In addition, we have
   $$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x) x \psi(x)\,dx \qquad (15.59)$$
   for the **average** position of the particle along the x-axis. This is often called an *expectation value*.

We want a function $g(p)$ that will give the same information about the momentum:

1. $g^*(p)g(p)\,dp$ is the probability density that our quantum particle has a momentum between p and $p+dp$.

2. $$\int_{-\infty}^{\infty} g^*(p)g(p)\,dp = 1. \tag{15.60}$$

3. $$\langle p\rangle = \int_{-\infty}^{\infty} g^*(p)\,p\,g(p)\,dp. \tag{15.61}$$

As subsequently shown, such a function is given by the Fourier transform of our space function $\psi(x)$. Specifically,[9]

$$g(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)e^{-ipx/\hbar}\,dx, \tag{15.62}$$

$$g^*(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi^*(x)e^{ipx/\hbar}\,dx. \tag{15.63}$$

The corresponding three-dimensional momentum function is

$$g(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\iiint_{-\infty}^{\infty}\psi(\mathbf{r})e^{-i\mathbf{r}\cdot\mathbf{p}/\hbar}\,d^3r.$$

To verify Eqs. (15.62) and (15.63), let us check on properties 2 and 3.

Property 2, the normalization, is automatically satisfied as a Parseval relation, Eq. (15.55). If the space function $\psi(x)$ is normalized to unity, the momentum function $g(p)$ is also normalized to unity.

To check on property 3, we must show that

$$\langle p\rangle = \int_{-\infty}^{\infty} g^*(p)pg(p)\,dp = \int_{-\infty}^{\infty}\psi^*(x)\frac{\hbar}{i}\frac{d}{dx}\psi(x)\,dx, \tag{15.64}$$

where $(\hbar/i)(d/dx)$ is the momentum operator in the space representation. We replace the momentum functions by Fourier-transformed space functions, and the first integral becomes

$$\frac{1}{2\pi\hbar}\iiint_{-\infty}^{\infty} pe^{-ip(x-x')/\hbar}\psi^*(x')\psi(x)\,dp\,dx'\,dx. \tag{15.65}$$

Now we use the plane-wave identity

$$pe^{-ip(x-x')/\hbar} = \frac{d}{dx}\left[-\frac{\hbar}{i}e^{-ip(x-x')/\hbar}\right], \tag{15.66}$$

[9]The $\hbar$ may be avoided by using the wave number k, $p = k\hbar$ (and $\mathbf{p} = \mathbf{k}\hbar$), so

$$\varphi(k) = \frac{1}{(2\pi)^{1/2}}\int\psi(x)e^{-ikx}\,dx.$$

An example of this notation appears in Section 16.1.

with p a constant, not an operator. Substituting into Eq. (15.65) and integrating by parts, holding x' and p constant, we obtain

$$\langle p\rangle = \iint\limits_{-\infty}^{\infty} \left[\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-ip(x-x')/\hbar}\,dp\right]\cdot\psi^*(x')\frac{\hbar}{i}\frac{d}{dx}\psi(x)\,dx'\,dx. \tag{15.67}$$

Here we assume $\psi(x)$ vanishes as $x \to \pm\infty$, eliminating the integrated part. Using the Dirac delta function, Eq. (15.21c), Eq. (15.67) reduces to Eq. (15.64) to verify our momentum representation.

Alternatively, if the integration over p is done first in Eq. (15.65), leading to

$$\int_{-\infty}^{\infty} pe^{-ip(x-x')/\hbar}\,dp = 2\pi i\hbar^2\delta'(x-x'),$$

and using Exercise 1.15.9, we can do the integration over x, which causes $\psi(x)$ to become $-d\psi(x')/dx'$. The remaining integral over x' is the right-hand side of Eq. (15.64).

Example 15.6.1 Hydrogen Atom

The hydrogen atom ground state[10] may be described by the spatial wave function

$$\psi(\mathbf{r}) = \left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0}, \tag{15.68}$$

a_0 being the Bohr radius, $4\pi\varepsilon_0\hbar^2/me^2$. We now have a three-dimensional wave function. The transform corresponding to Eq. (15.62) is

$$g(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\int \psi(\mathbf{r})e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\,d^3r. \tag{15.69}$$

Substituting Eq. (15.68) into Eq. (15.69) and using

$$\int e^{-ar+i\mathbf{b}\cdot\mathbf{r}}\,d^3r = \frac{8\pi a}{(a^2+b^2)^2}, \tag{15.70}$$

we obtain the hydrogenic momentum wave function,

$$g(\mathbf{p}) = \frac{2^{3/2}}{\pi}\frac{a_0^{3/2}\hbar^{5/2}}{(a_0^2p^2+\hbar^2)^2}. \tag{15.71}$$

Such momentum functions have been found useful in problems like Compton scattering from atomic electrons, the wavelength distribution of the scattered radiation, depending on the momentum distribution of the target electrons.

The relation between the ordinary space representation and the momentum representation may be clarified by considering the basic commutation relations of quantum mechanics. We go from a classical Hamiltonian to the Schrödinger wave equation by requiring that momentum p and position x **not** commute. Instead, we require that

$$[p,x] \equiv px - xp = -i\hbar. \tag{15.72}$$

[10]See E. V. Ivash, A momentum representation treatment of the hydrogen atom problem. *Am. J. Phys.* **40**: 1095 (1972) for a momentum representation treatment of the hydrogen atom $l=0$ states.

For the multidimensional case, Eq. (15.72) is replaced by

$$[p_i, x_j] = -i\hbar\delta_{ij}. \tag{15.73}$$

The Schrödinger (space) representation is obtained by using

$$x \to x: \quad p_i \to -i\hbar\frac{\partial}{\partial x_i},$$

replacing the momentum by a partial space derivative. We see that

$$[p, x]\psi(x) = -i\hbar\psi(x). \tag{15.74}$$

However, Eq. (15.72) can equally well be satisfied by using

$$p \to p: \quad x_j \to i\hbar\frac{\partial}{\partial p_j}.$$

This is the momentum representation. Then

$$[p, x]g(p) = -i\hbar g(p). \tag{15.75}$$

Hence the representation (x) is not unique; (p) is an alternate possibility.

In general, the Schrödinger representation (x) leading to the Schrödinger wave equation is more convenient because the potential energy V is generally given as a function of position $V(x, y, z)$. The momentum representation (p) usually leads to an integral equation (compare Chapter 16 for the pros and cons of the integral equations). For an exception, consider the harmonic oscillator. ■

Example 15.6.2 Harmonic Oscillator

The classical Hamiltonian (kinetic energy + potential energy = total energy) is

$$H(p, x) = \frac{p^2}{2m} + \frac{1}{2}kx^2 = E, \tag{15.76}$$

where k is the Hooke's law constant.

In the Schrödinger representation we obtain

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}kx^2\psi(x) = E\psi(x). \tag{15.77}$$

For total energy E equal to $\sqrt{(k/m)}\hbar/2$ there is an unnormalized solution (Section 13.1),

$$\psi(x) = e^{-(\sqrt{mk}/2\hbar)x^2}. \tag{15.78}$$

The momentum representation leads to

$$\frac{p^2}{2m}g(p) - \frac{\hbar^2 k}{2}\frac{d^2 g(p)}{dp^2} = Eg(p). \tag{15.79}$$

Again, for

$$E = \sqrt{\frac{k}{m}}\frac{\hbar}{2} \tag{15.80}$$

the momentum wave equation (15.79) is satisfied by the unnormalized

$$g(p) = e^{-p^2/(2\hbar\sqrt{mk})}. \tag{15.81}$$

Either representation, space or momentum (and an infinite number of other possibilities), may be used, depending on which is more convenient for the particular problem under attack.

The demonstration that $g(p)$ is the momentum wave function corresponding to Eq. (15.78) — that it is the Fourier inverse transform of Eq. (15.78) — is left as Exercise 15.6.3. ■

Exercises

15.6.1 The function $e^{i\mathbf{k}\cdot\mathbf{r}}$ describes a plane wave of momentum $\mathbf{p} = \hbar\mathbf{k}$ normalized to unit density. (Time dependence of $e^{-i\omega t}$ is assumed.) Show that these plane-wave functions satisfy an orthogonality relation

$$\int \left(e^{i\mathbf{k}\cdot\mathbf{r}}\right)^* e^{i\mathbf{k}'\cdot\mathbf{r}} dx\,dy\,dz = (2\pi)^3\delta(\mathbf{k} - \mathbf{k}').$$

15.6.2 An infinite plane wave in quantum mechanics may be represented by the function

$$\psi(x) = e^{ip'x/\hbar}.$$

Find the corresponding momentum distribution function. Note that it has an infinity and that $\psi(x)$ is not normalized.

15.6.3 A linear quantum oscillator in its ground state has a wave function

$$\psi(x) = a^{-1/2}\pi^{-1/4}e^{-x^2/2a^2}.$$

Show that the corresponding momentum function is

$$g(p) = a^{1/2}\pi^{-1/4}\hbar^{-1/2}e^{-a^2p^2/2\hbar^2}.$$

15.6.4 The nth excited state of the linear quantum oscillator is described by

$$\psi_n(x) = a^{-1/2}2^{-n/2}\pi^{-1/4}(n!)^{-1/2}e^{-x^2/2a^2}H_n(x/a),$$

where $H_n(x/a)$ is the nth Hermite polynomial, Section 13.1. As an extension of Exercise 15.6.3, find the momentum function corresponding to $\psi_n(x)$.
Hint. $\psi_n(x)$ may be represented by $(\hat{a}^\dagger)^n\psi_0(x)$, where $\hat{a}^\dagger$ is the raising operator, Exercise 13.1.14 to 13.1.16.

15.6.5 A free particle in quantum mechanics is described by a plane wave

$$\psi_k(x,t) = e^{i[kx-(\hbar k^2/2m)t]}.$$

Combining waves of adjacent momentum with an amplitude weighting factor $\varphi(k)$, we form a wave packet

$$\Psi(x,t) = \int_{-\infty}^{\infty} \varphi(k)e^{i[kx-(\hbar k^2/2m)t]}\,dk.$$

(a) Solve for $\varphi(k)$ given that

$$\Psi(x, 0) = e^{-x^2/2a^2}.$$

(b) Using the known value of $\varphi(k)$, integrate to get the explicit form of $\Psi(x, t)$. Note that this wave packet diffuses or spreads out with time.

$$\text{ANS. } \Psi(x, t) = \frac{e^{-\{x^2/2[a^2+(i\hbar/m)t]\}}}{[1 + (i\hbar t/ma^2)]^{1/2}}.$$

Note. An interesting discussion of this problem from the evolution operator point of view is given by S. M. Blinder, Evolution of a Gaussian wave packet, *Am. J. Phys.* **36**: 525 (1968).

15.6.6 Find the time-dependent momentum wave function $g(k, t)$ corresponding to $\Psi(x, t)$ of Exercise 15.6.5. Show that the momentum wave packet $g^*(k, t)g(k, t)$ is **independent** of time.

15.6.7 The deuteron, Example 10.1.2, may be described reasonably well with a Hulthén wave function

$$\psi(\mathbf{r}) = A[e^{-\alpha r} - e^{-\beta r}]/r,$$

with A, α, and β constants. Find $g(\mathbf{p})$, the corresponding momentum function.
Note. The Fourier transform may be rewritten as Fourier sine and cosine transforms or as a Laplace transform, Section 15.8.

15.6.8 The nuclear form factor $F(k)$ and the charge distribution $\rho(r)$ are three-dimensional Fourier transforms of each other:

$$F(k) = \frac{1}{(2\pi)^{3/2}} \int \rho(r) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3r.$$

If the measured form factor is

$$F(k) = (2\pi)^{-3/2}\left(1 + \frac{k^2}{a^2}\right)^{-1},$$

find the corresponding charge distribution.

$$\text{ANS. } \rho(r) = \frac{a^2}{4\pi}\frac{e^{-ar}}{r}.$$

15.6.9 Check the normalization of the hydrogen momentum wave function

$$g(\mathbf{p}) = \frac{2^{3/2}}{\pi}\frac{a_0^{3/2}\hbar^{5/2}}{(a_0^2 p^2 + \hbar^2)^2}$$

by direct evaluation of the integral

$$\int g^*(\mathbf{p}) g(\mathbf{p})\, d^3p.$$

15.6.10 With $\psi(\mathbf{r})$ a wave function in ordinary space and $\varphi(\mathbf{p})$ the corresponding momentum function, show that

(a) $\dfrac{1}{(2\pi\hbar)^{3/2}}\displaystyle\int \mathbf{r}\psi(\mathbf{r})e^{-i\mathbf{r}\cdot\mathbf{p}/\hbar}\,d^3r = i\hbar\,\nabla_p\varphi(\mathbf{p}),$

(b) $\dfrac{1}{(2\pi\hbar)^{3/2}}\displaystyle\int \mathbf{r}^2\psi(\mathbf{r})e^{-\mathbf{r}\cdot\mathbf{p}/\hbar}\,d^3r = (i\hbar\,\nabla_p)^2\varphi(\mathbf{p}).$

Note. ∇_p is the gradient in momentum space:

$$\hat{\mathbf{x}}\frac{\partial}{\partial p_x} + \hat{\mathbf{y}}\frac{\partial}{\partial p_y} + \hat{\mathbf{z}}\frac{\partial}{\partial p_z}.$$

These results may be extended to any positive integer power of r and therefore to any (analytic) function that may be expanded as a Maclaurin series in r.

15.6.11 The ordinary space wave function $\psi(\mathbf{r}, t)$ satisfies the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r})\psi.$$

Show that the corresponding time-dependent momentum wave function satisfies the analogous equation,

$$i\hbar\frac{\partial\varphi(\mathbf{p}, t)}{\partial t} = \frac{p^2}{2m}\varphi + V(i\hbar\,\nabla_p)\varphi.$$

Note. Assume that $V(\mathbf{r})$ may be expressed by a Maclaurin series and use Exercise 15.6.10. $V(i\hbar\,\nabla_p)$ is the same function of the variable $i\hbar\,\nabla_p$ that $V(\mathbf{r})$ is of the variable $\mathbf{r}$.

15.6.12 The one-dimensional time-independent Schrödinger wave equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

For the special case of $V(x)$ an analytic function of x, show that the corresponding momentum wave equation is

$$V\left(i\hbar\frac{d}{dp}\right)g(p) + \frac{p^2}{2m}g(p) = Eg(p).$$

Derive this momentum wave equation from the Fourier transform, Eq. (15.62), and its inverse. Do not use the substitution $x \to i\hbar(d/dp)$ directly.

15.7 TRANSFER FUNCTIONS

A time-dependent electrical pulse may be regarded as built-up as a superposition of plane waves of many frequencies. For angular frequency ω we have a contribution

$$F(\omega)e^{i\omega t}.$$

Then the complete pulse may be written as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega. \tag{15.82}$$

FIGURE 15.6 Servomechanism or a stereo amplifier.

Because the angular frequency ω is related to the linear frequency ν by

$$\nu = \frac{\omega}{2\pi},$$

it is customary to associate the entire $1/2\pi$ factor with this integral.

But if ω is a frequency, what about the negative frequencies? The negative ω may be looked on as a mathematical device to avoid dealing with two functions ($\cos\omega t$ and $\sin\omega t$) separately (compare Section 14.1).

Because Eq. (15.82) has the form of a Fourier transform, we may solve for $F(\omega)$ by writing the inverse transform,

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt. \tag{15.83}$$

Equation (15.83) represents a **resolution of the pulse** $f(t)$ into its angular frequency components. Equation (15.82) is a **synthesis of the pulse** from its components.

Consider some device, such as a servomechanism or a stereo amplifier (Fig. 15.6), with an input $f(t)$ and an output $g(t)$. For an input of a single frequency ω, $f_\omega(t) = e^{i\omega t}$, the amplifier will alter the amplitude and may also change the phase. The changes will probably depend on the frequency. Hence

$$g_\omega(t) = \varphi(\omega) f_\omega(t). \tag{15.84}$$

This amplitudes- and phase-modifying function $\varphi(\omega)$ is called a **transfer** function. It usually will be complex:

$$\varphi(\omega) = u(\omega) + i v(\omega), \tag{15.85}$$

where the functions $u(\omega)$ and $v(\omega)$ are real.

In Eq. (15.84) we assume that the transfer function $\varphi(\omega)$ is independent of input amplitude and of the presence or absence of any other frequency components. That is, we are assuming a linear mapping of $f(t)$ onto $g(t)$. Then the total output may be obtained by integrating over the entire input, as modified by the amplifier

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\omega) F(\omega) e^{i\omega t} d\omega. \tag{15.86}$$

The transfer function is characteristic of the amplifier. Once the transfer function is known (measured or calculated), the output $g(t)$ can be calculated for any input $f(t)$. Let us consider $\varphi(\omega)$ as the Fourier (inverse) transform of some function $\Phi(t)$:

$$\varphi(\omega) = \int_{-\infty}^{\infty} \Phi(t) e^{-i\omega t} dt. \tag{15.87}$$

Then Eq. (15.86) is the Fourier transform of two inverse transforms. From Section 15.5 we obtain the convolution

$$g(t) = \int_{-\infty}^{\infty} f(\tau)\Phi(t-\tau)\,d\tau. \tag{15.88}$$

Interpreting Eq. (15.88), we have an input — a "cause" — $f(\tau)$, modified by $\Phi(t-\tau)$, producing an output — an "effect" — $g(t)$. Adopting the concept of **causality** — that the cause precedes the effect — we must require $\tau < t$. We do this by requiring

$$\Phi(t-\tau) = 0, \qquad \tau > t. \tag{15.89}$$

Then Eq. (15.88) becomes

$$g(t) = \int_{-\infty}^{t} f(\tau)\Phi(t-\tau)\,d\tau. \tag{15.90}$$

The adoption of Eq. (15.89) has profound consequences here and equivalently in dispersion theory, Section 7.2.

Significance of $\Phi(t)$

To see the significance of Φ, let $f(\tau)$ be a sudden impulse starting at $\tau = 0$,

$$f(\tau) = \delta(\tau),$$

where $\delta(\tau)$ is a Dirac delta distribution on the positive side of the origin. Then Eq. (15.90) becomes

$$\begin{aligned} g(t) &= \int_{-\infty}^{t} \delta(\tau)\Phi(t-\tau)\,d\tau, \\ g(t) &= \begin{cases} \Phi(t), & t > 0, \\ 0, & t < 0. \end{cases} \end{aligned} \tag{15.91}$$

This identifies $\Phi(t)$ as the output function corresponding to a unit impulse at $t = 0$. Equation (15.91) also serves to establish that $\Phi(t)$ is real. Our original transfer function gives the steady-state output corresponding to a unit-amplitude single-frequency input. $\Phi(t)$ and $\varphi(\omega)$ are Fourier transforms of each other.

From Eq. (15.87) we now have

$$\varphi(\omega) = \int_{0}^{\infty} \Phi(t)e^{-i\omega t}\,dt, \tag{15.92}$$

with the lower limit set equal to zero by causality (Eq. (15.89)). With $\Phi(t)$ real from Eq. (15.91) we separate real and imaginary parts and write

$$\begin{aligned} u(\omega) &= \int_{0}^{\infty} \Phi(t)\cos\omega t\,dt, \\ v(\omega) &= -\int_{0}^{\infty} \Phi(t)\sin\omega t\,dt, \qquad \omega > 0. \end{aligned} \tag{15.93}$$

From this we see that the real part of $\varphi(\omega), u(\omega)$, is even, whereas the imaginary part of $\varphi(\omega), v(\omega)$, is odd:

$$u(-\omega) = u(\omega), \qquad v(-\omega) = -v(\omega).$$

Compare this result with Exercise 15.3.1.

Interpreting Eq. (15.93) as Fourier cosine and sine transforms, we have

$$\begin{aligned}\Phi(t) &= \frac{2}{\pi}\int_0^\infty u(\omega)\cos\omega t\,d\omega \\ &= -\frac{2}{\pi}\int_0^\infty v(\omega)\sin\omega t\,d\omega, \qquad t > 0.\end{aligned} \tag{15.94}$$

Combining Eqs. (15.93) and (15.94), we obtain

$$v(\omega) = -\int_0^\infty \sin\omega t\left\{\frac{2}{\pi}\int_0^\infty u(\omega')\cos\omega' t\,d\omega'\right\}dt, \tag{15.95}$$

showing that if our transfer function has a real part, it will also have an imaginary part (and vice versa). Of course, this assumes that the Fourier transforms exist, thus excluding cases such as $\Phi(t) = 1$.

The imposition of causality has led to a mutual interdependence of the real and imaginary parts of the transfer function. The reader should compare this with the results of the dispersion theory of Section 7.2, also involving causality.

It may be helpful to show that the parity properties of $u(\omega)$ and $v(\omega)$ require $\Phi(t)$ to vanish for negative t. Inverting Eq. (15.87), we have

$$\Phi(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \big[u(\omega) + i v(\omega)\big]\big[\cos\omega t + i\sin\omega t\big]\,d\omega. \tag{15.96}$$

With $u(\omega)$ even and $v(\omega)$ odd, Eq. (15.96) becomes

$$\Phi(t) = \frac{1}{\pi}\int_0^\infty u(\omega)\cos\omega t\,d\omega - \frac{1}{\pi}\int_0^\infty v(\omega)\sin\omega t\,d\omega. \tag{15.97}$$

From Eq. (15.94),

$$\int_0^\infty u(\omega)\cos\omega t\,d\omega = -\int_0^\infty v(\omega)\sin\omega t\,d\omega, \quad t > 0. \tag{15.98}$$

If we reverse the sign of t, $\sin\omega t$ reverses sign and, from Eq. (15.97),

$$\Phi(t) = 0, \qquad t < 0$$

(demonstrating the internal consistency of our analysis).

Exercise

15.7.1 Derive the convolution

$$g(t) = \int_{-\infty}^\infty f(\tau)\Phi(t-\tau)\,d\tau.$$

15.8 LAPLACE TRANSFORMS

Definition

The Laplace transform $f(s)$ or $\mathcal{L}$ of a function $F(t)$ is defined by[11]

$$f(s) = \mathcal{L}\{F(t)\} = \lim_{a\to\infty}\int_0^a e^{-st}F(t)\,dt = \int_0^\infty e^{-st}F(t)\,dt. \tag{15.99}$$

A few comments on the existence of the integral are in order. The infinite integral of $F(t)$,

$$\int_0^\infty F(t)\,dt,$$

need not exist. For instance, $F(t)$ may diverge exponentially for large t. However, if there is some constant s_0 such that

$$\left|e^{-s_0 t}F(t)\right| \le M, \tag{15.100}$$

a positive constant for sufficiently large t, $t > t_0$, the Laplace transform (Eq. (15.99)) will exist for $s > s_0$; $F(t)$ is said to be of *exponential order*. As a counterexample, $F(t) = e^{t^2}$ does not satisfy the condition given by Eq. (15.100) and is **not** of exponential order. $\mathcal{L}\{e^{t^2}\}$ does **not** exist.

The Laplace transform may also fail to exist because of a sufficiently strong singularity in the function $F(t)$ as $t \to 0$; that is,

$$\int_0^\infty e^{-st}t^n\,dt$$

diverges at the origin for $n \le -1$. The Laplace transform $\mathcal{L}\{t^n\}$ does not exist for $n \le -1$.

Since, for two functions $F(t)$ and $G(t)$, for which the integrals exist

$$\mathcal{L}\{aF(t) + bG(t)\} = a\mathcal{L}\{F(t)\} + b\mathcal{L}\{G(t)\}, \tag{15.101}$$

the operation denoted by $\mathcal{L}$ is **linear**.

Elementary Functions

To introduce the Laplace transform, let us apply the operation to some of the elementary functions. In all cases we assume that $F(t) = 0$ for $t < 0$. If

$$F(t) = 1, \qquad t > 0,$$

[11] This is sometimes called a **one-sided Laplace transform**; the integral from $-\infty$ to $+\infty$ is referred to as a **two-sided Laplace transform**. Some authors introduce an additional factor of s. This extra s appears to have little advantage and continually gets in the way (compare Jeffreys and Jeffreys, Section 14.13 — see the Additional Readings — for additional comments). Generally, we take s to be real and positive. It is possible to have s complex, provided $\Re(s) > 0$.

then

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \frac{1}{s}, \qquad \text{for } s > 0. \tag{15.102}$$

Again, let

$$F(t) = e^{kt}, \qquad t > 0.$$

The Laplace transform becomes

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st} e^{kt}\,dt = \frac{1}{s-k}, \qquad \text{for } s > k. \tag{15.103}$$

Using this relation, we obtain the Laplace transform of certain other functions. Since

$$\cosh kt = \frac{1}{2}\left(e^{kt} + e^{-kt}\right), \qquad \sinh kt = \frac{1}{2}\left(e^{kt} - e^{-kt}\right), \tag{15.104}$$

we have

$$\begin{aligned}\mathcal{L}\{\cosh kt\} &= \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \frac{s}{s^2-k^2},\\ \mathcal{L}\{\sinh kt\} &= \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) = \frac{k}{s^2-k^2},\end{aligned} \tag{15.105}$$

both valid for $s > k$. We have the relations

$$\cos kt = \cosh ikt, \qquad \sin kt = -i \sinh ikt. \tag{15.106}$$

Using Eqs. (15.105) with k replaced by ik, we find that the Laplace transforms are

$$\begin{aligned}\mathcal{L}\{\cos kt\} &= \frac{s}{s^2+k^2},\\ \mathcal{L}\{\sin kt\} &= \frac{k}{s^2+k^2},\end{aligned} \tag{15.107}$$

both valid for $s > 0$. Another derivation of this last transform is given in the next section. Note that $\lim_{s\to 0} \mathcal{L}\{\sin kt\} = 1/k$. The Laplace transform assigns a value of $1/k$ to $\int_0^\infty \sin kt\,dt$.

Finally, for $F(t) = t^n$, we have

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n\,dt,$$

which is just the factorial function. Hence

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad s > 0,\ n > -1. \tag{15.108}$$

Note that in all these transforms we have the variable s in the denominator — negative powers of s. In particular, $\lim_{s\to\infty} f(s) = 0$. The significance of this point is that if $f(s)$ involves positive powers of s ($\lim_{s\to\infty} f(s) \to \infty$), then no inverse transform exists.

Inverse Transform

There is little importance to these operations unless we can carry out the inverse transform, as in Fourier transforms. That is, with

$$\mathcal{L}\{F(t)\} = f(s),$$

then

$$\mathcal{L}^{-1}\{f(s)\} = F(t). \tag{15.109}$$

This inverse transform is **not** unique. Two functions $F_1(t)$ and $F_2(t)$ may have the same transform, $f(s)$. However, in this case

$$F_1(t) - F_2(t) = N(t),$$

where $N(t)$ is a null function (Fig. 15.7), indicating that

$$\int_0^{t_0} N(t)\,dt = 0,$$

for all positive t_0. This result is known as **Lerch's theorem**. Therefore to the physicist and engineer $N(t)$ may almost always be taken as zero and the inverse operation becomes unique.

The inverse transform can be determined in various ways. (1) A table of transforms can be built up and used to carry out the inverse transformation, exactly as a table of logarithms can be used to look up antilogarithms. The preceding transforms constitute the beginnings of such a table. For a more complete set of Laplace transforms see upcoming Table 15.2 or AMS-55, Chapter 29 (see footnote 4 in Chapter 5 for the reference). Employing partial fraction expansions and various operational theorems, which are considered in succeeding sections, facilitates use of the tables.

- There is some justification for suspecting that these tables are probably of more value in solving textbook exercises than in solving real-world problems.
- (2) A general technique for $\mathcal{L}^{-1}$ will be developed in Section 15.12 by using the calculus of residues.

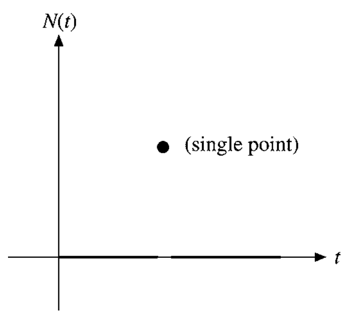

FIGURE 15.7 A possible null function.

- (3) For the difficulties and the possibilities of a numerical approach — numerical inversion — we refer to the Additional Readings.

Partial Fraction Expansion

Utilization of a table of transforms (or inverse transforms) is facilitated by expanding $f(s)$ in **partial fractions**.

Frequently $f(s)$, our transform, occurs in the form $g(s)/h(s)$, where $g(s)$ and $h(s)$ are polynomials with no common factors, $g(s)$ being of lower degree than $h(s)$. If the factors of $h(s)$ are all linear and distinct, then by the method of partial fractions we may write

$$f(s) = \frac{c_1}{s-a_1} + \frac{c_2}{s-a_2} + \cdots + \frac{c_n}{s-a_n}, \tag{15.110}$$

where the c_i are independent of s. The a_i are the roots of $h(s)$. If any one of the roots, say, a_1, is multiple (occurring m times), then $f(s)$ has the form

$$f(s) = \frac{c_{1,m}}{(s-a_1)^m} + \frac{c_{1,m-1}}{(s-a_1)^{m-1}} + \cdots + \frac{c_{1,1}}{s-a_1} + \sum_{i=2}^{n} \frac{c_i}{s-a_i}. \tag{15.111}$$

Finally, if one of the factors is quadratic, $(s^2 + ps + q)$, then the numerator, instead of being a simple constant, will have the form

$$\frac{as+b}{s^2+ps+q}.$$

There are various ways of determining the constants introduced. For instance, in Eq. (15.110) we may multiply through by $(s-a_i)$ and obtain

$$c_i = \lim_{s \to a_i} (s-a_i) f(s). \tag{15.112}$$

In elementary cases a direct solution is often the easiest.

Example 15.8.1 PARTIAL FRACTION EXPANSION

Let

$$f(s) = \frac{k^2}{s(s^2+k^2)} = \frac{c}{s} + \frac{as+b}{s^2+k^2}. \tag{15.113}$$

Putting the right side of the equation over a common denominator and equating like powers of s in the numerator, we obtain

$$\frac{k^2}{s(s^2+k^2)} = \frac{c(s^2+k^2) + s(as+b)}{s(s^2+k^2)}, \tag{15.114}$$

$$c + a = 0, \quad s^2; \qquad b = 0, \quad s^1; \qquad ck^2 = k^2, \quad s^0.$$

Solving these $(s \neq 0)$, we have

$$c = 1, \qquad b = 0, \qquad a = -1,$$

giving

$$f(s) = \frac{1}{s} - \frac{s}{s^2 + k^2}, \tag{15.115}$$

and

$$\mathcal{L}^{-1}\{f(s)\} = 1 - \cos kt \tag{15.116}$$

by Eqs. (15.102) and (15.106). ■

Example 15.8.2 A Step Function

As one application of Laplace transforms, consider the evaluation of

$$F(t) = \int_0^\infty \frac{\sin tx}{x} dx. \tag{15.117}$$

Suppose we take the Laplace transform of this definite (and improper) integral:

$$\mathcal{L}\left\{\int_0^\infty \frac{\sin tx}{x} dx\right\} = \int_0^\infty e^{-st} \int_0^\infty \frac{\sin tx}{x} dx\, dt. \tag{15.118}$$

Now, interchanging the order of integration (which is justified),[12] we get

$$\int_0^\infty \frac{1}{x}\left[\int_0^\infty e^{-st} \sin tx\, dt\right] dx = \int_0^\infty \frac{dx}{s^2 + x^2}, \tag{15.119}$$

since the factor in square brackets is just the Laplace transform of $\sin tx$. From the integral tables,

$$\int_0^\infty \frac{dx}{s^2 + x^2} = \frac{1}{s} \tan^{-1}\left(\frac{x}{s}\right)\Big|_0^\infty = \frac{\pi}{2s} = f(s). \tag{15.120}$$

By Eq. (15.102) we carry out the inverse transformation to obtain

$$F(t) = \frac{\pi}{2}, \qquad t > 0, \tag{15.121}$$

in agreement with an evaluation by the calculus of residues (Section 7.1). It has been assumed that $t > 0$ in $F(t)$. For $F(-t)$ we need note only that $\sin(-tx) = -\sin tx$, giving $F(-t) = -F(t)$. Finally, if $t = 0$, $F(0)$ is clearly zero. Therefore

$$\int_0^\infty \frac{\sin tx}{x} dx = \frac{\pi}{2}\left[2u(t) - 1\right] = \begin{cases} \frac{\pi}{2}, & t > 0 \\ 0, & t = 0 \\ -\frac{\pi}{2}, & t < 0. \end{cases} \tag{15.122}$$

Note that $\int_0^\infty (\sin tx/x)\, dx$, taken as a function of t, describes a step function (Fig. 15.8), a step of height π at $t = 0$. This is consistent with Eq. (1.174). ■

The technique in the preceding example was to (1) introduce a second integration—the Laplace transform, (2) reverse the order of integration and integrate, and (3) take the

[12] See—in the Additional Readings—Jeffreys and Jeffreys (1966), Chapter 1 (uniform convergence of integrals).

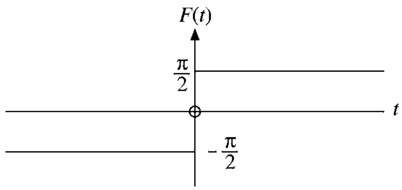

FIGURE 15.8 $F(t) = \int_0^\infty \frac{\sin tx}{x} dx$, a step function.

inverse Laplace transform. There are many opportunities where this technique of reversing the order of integration can be applied and proved useful. Exercise 15.8.6 is a variation of this.

Exercises

15.8.1 Prove that

$$\lim_{s\to\infty} sf(s) = \lim_{t\to+0} F(t).$$

Hint. Assume that $F(t)$ can be expressed as $F(t) = \sum_{n=0}^{\infty} a_n t^n$.

15.8.2 Show that

$$\frac{1}{\pi} \lim_{s\to 0} \mathcal{L}\{\cos xt\} = \delta(x).$$

15.8.3 Verify that

$$\mathcal{L}\left\{\frac{\cos at - \cos bt}{b^2 - a^2}\right\} = \frac{s}{(s^2+a^2)(s^2+b^2)}, \qquad a^2 \neq b^2.$$

15.8.4 Using partial fraction expansions, show that

(a) $$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = \frac{e^{-at} - e^{-bt}}{b-a}, \qquad a \neq b.$$

(b) $$\mathcal{L}^{-1}\left\{\frac{s}{(s+a)(s+b)}\right\} = \frac{ae^{-at} - be^{-bt}}{a-b}, \qquad a \neq b.$$

15.8.5 Using partial fraction expansions, show that for $a^2 \neq b^2$,

(a) $$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = -\frac{1}{a^2-b^2}\left\{\frac{\sin at}{a} - \frac{\sin bt}{b}\right\},$$

(b) $$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{a^2-b^2}\{a \sin at - b \sin bt\}.$$

15.8.6 The electrostatic potential of a charged conducting disk is known to have the general form (circular cylindrical coordinates)

$$\Phi(\rho, z) = \int_0^\infty e^{-k|z|} J_0(k\rho) f(k)\, dk,$$

with $f(k)$ unknown. At large distances ($z \to \infty$) the potential must approach the Coulomb potential $Q/4\pi\varepsilon_0 z$. Show that

$$\lim_{k\to 0} f(k) = \frac{q}{4\pi\varepsilon_0}.$$

Hint. You may set $\rho = 0$ and assume a Maclaurin expansion of $f(k)$ or, using e^{-kz}, construct a delta sequence.

15.8.7 Show that

(a) $$\int_0^\infty \frac{\cos s}{s^\nu} ds = \frac{\pi}{2(\nu-1)!\cos(\nu\pi/2)}, \qquad 0 < \nu < 1,$$

(b) $$\int_0^\infty \frac{\sin s}{s^\nu} ds = \frac{\pi}{2(\nu-1)!\sin(\nu\pi/2)}, \qquad 0 < \nu < 2,$$

Why is ν restricted to (0, 1) for (a), to (0, 2) for (b)? These integrals may be interpreted as Fourier transforms of $s^{-\nu}$ and as Mellin transforms of $\sin s$ and $\cos s$.
Hint. Replace $s^{-\nu}$ by a Laplace transform integral: $\mathcal{L}\{t^{\nu-1}\}/(\nu-1)!$. Then integrate with respect to s. The resulting integral can be treated as a beta function (Section 8.4).

15.8.8 A function $F(t)$ can be expanded in a power series (Maclaurin); that is,

$$F(t) = \sum_{n=0}^\infty a_n t^n.$$

Then

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} \sum_{n=0}^\infty a_n t^n dt = \sum_{n=0}^\infty a_n \int_0^\infty e^{-st} t^n dt.$$

Show that $f(s)$, the Laplace transform of $F(t)$, contains no powers of s greater than s^{-1}. Check your result by calculating $\mathcal{L}\{\delta(t)\}$, and comment on this fiasco.

15.8.9 Show that the Laplace transform of $M(a, c, x)$ is

$$\mathcal{L}\{M(a, c, x)\} = \frac{1}{s}\,{}_2F_1\left(a, 1; c, \frac{1}{s}\right).$$

15.9 Laplace Transform of Derivatives

Perhaps the main application of Laplace transforms is in converting differential equations into simpler forms that may be solved more easily. It will be seen, for instance, that coupled differential equations with constant coefficients transform to simultaneous linear algebraic equations.

Let us transform the first derivative of $F(t)$:

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} \frac{dF(t)}{dt} dt.$$

Integrating by parts, we obtain

$$\begin{aligned}\mathcal{L}\{F'(t)\} &= e^{-st} F(t)\Big|_0^\infty + s\int_0^\infty e^{-st} F(t)\,dt \\ &= s\mathcal{L}\{F(t)\} - F(0). \end{aligned} \tag{15.123}$$

Strictly speaking, $F(0) = F(+0)$[13] and dF/dt is required to be at least piecewise continuous for $0 \le t < \infty$. Naturally, both $F(t)$ and its derivative must be such that the integrals do not diverge. Incidentally, Eq. (15.123) provides another proof of Exercise 15.8.8. An extension gives

$$\mathcal{L}\{F^{(2)}(t)\} = s^2\mathcal{L}\{F(t)\} - sF(+0) - F'(+0), \tag{15.124}$$

$$\mathcal{L}\{F^{(n)}(t)\} = s^n\mathcal{L}\{F(t)\} - s^{n-1}F(+0) - \cdots - F^{(n-1)}(+0). \tag{15.125}$$

The Laplace transform, like the Fourier transform, replaces differentiation with multiplication. In the following examples ODEs become algebraic equations. Here is the power and the utility of the Laplace transform. But see Example 15.10.3 for what may happen if the coefficients are not constant.

Note how the initial conditions, $F(+0)$, $F'(+0)$, and so on, are incorporated into the transform. Equation (15.124) may be used to derive $\mathcal{L}\{\sin kt\}$. We use the identity

$$-k^2 \sin kt = \frac{d^2}{dt^2} \sin kt. \tag{15.126}$$

Then applying the Laplace transform operation, we have

$$\begin{aligned}-k^2\mathcal{L}\{\sin kt\} &= \mathcal{L}\left\{\frac{d^2}{dt^2}\sin kt\right\} \\ &= s^2\mathcal{L}\{\sin kt\} - s\sin(0) - \frac{d}{dt}\sin kt\Big|_{t=0}. \end{aligned} \tag{15.127}$$

Since $\sin(0) = 0$ and $d/dt \sin kt|_{t=0} = k$,

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2}, \tag{15.128}$$

verifying Eq. (15.107).

[13] Zero is approached from the positive side.

Example 15.9.1 SIMPLE HARMONIC OSCILLATOR

As a physical example, consider a mass m oscillating under the influence of an ideal spring, spring constant k. As usual, friction is neglected. Then Newton's second law becomes

$$m\frac{d^2X(t)}{dt^2}+kX(t)=0; \tag{15.129}$$

also, we take as initial conditions

$$X(0)=X_0, \qquad X'(0)=0.$$

Applying the Laplace transform, we obtain

$$m\mathcal{L}\left\{\frac{d^2X}{dt^2}\right\}+k\mathcal{L}\big\{X(t)\big\}=0, \tag{15.130}$$

and by use of Eq. (15.124) this becomes

$$ms^2x(s)-msX_0+kx(s)=0, \tag{15.131}$$

$$x(s)=X_0\frac{s}{s^2+\omega_0^2}, \qquad \text{with } \omega_0^2\equiv\frac{k}{m}. \tag{15.132}$$

From Eq. (15.107) this is seen to be the transform of $\cos\omega_0 t$, which gives

$$X(t)=X_0\cos\omega_0 t, \tag{15.133}$$

as expected. ■

Example 15.9.2 EARTH'S NUTATION

A somewhat more involved example is the nutation of the earth's poles (force-free precession). If we treat the Earth as a rigid (oblate) spheroid, the Euler equations of motion reduce to

$$\frac{dX}{dt}=-aY, \qquad \frac{dY}{dt}=+aX, \tag{15.134}$$

where $a\equiv[(I_z-I_x)/I_z]\omega_z$, $X=\omega_x$, $Y=\omega_y$ with angular velocity vector $\boldsymbol{\omega}=(\omega_x,\omega_y,\omega_z)$ (Fig. 15.9), $I_z=$ moment of inertia about the z-axis and $I_y=I_x$ moment of inertia about the x- (or y-)axis. The z-axis coincides with the axis of symmetry of the Earth. It differs from the axis for the Earth's daily rotation, $\boldsymbol{\omega}$, by some 15 meters, measured at the poles. Transformation of these coupled differential equations yields

$$sx(s)-X(0)=-ay(s), \qquad sy(s)-Y(0)=ax(s). \tag{15.135}$$

Combining to eliminate $y(s)$, we have

$$s^2x(s)-sX(0)+aY(0)=-a^2x(s),$$

or

$$x(s)=X(0)\frac{s}{s^2+a^2}-Y(0)\frac{a}{s^2+a^2}. \tag{15.136}$$

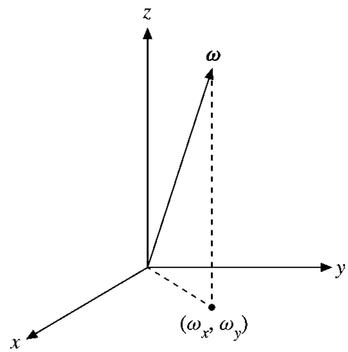

FIGURE 15.9

Hence

$$X(t) = X(0)\cos at - Y(0)\sin at. \tag{15.137}$$

Similarly,

$$Y(t) = X(0)\sin at + Y(0)\cos at. \tag{15.138}$$

This is seen to be a rotation of the vector (X, Y) counterclockwise (for $a > 0$) about the z-axis with angle $\theta = at$ and angular velocity a.

A direct interpretation may be found by choosing the time axis so that $Y(0) = 0$. Then

$$X(t) = X(0)\cos at, \qquad Y(t) = X(0)\sin at, \tag{15.139}$$

which are the parametric equations for rotation of (X, Y) in a circular orbit of radius $X(0)$, with angular velocity a in the counterclockwise sense.

In the case of the Earth's angular velocity, vector $X(0)$ is about 15 meters, whereas a, as defined here, corresponds to a period $(2\pi/a)$ of some 300 days. Actually because of departures from the idealized rigid body assumed in setting up Euler's equations, the period is about 427 days.[14] If in Eq. (15.134) we set

$$X(t) = L_x, \qquad Y(t) = L_y,$$

where L_x and L_y are the x- and y-components of the angular momentum **L**, $a = -g_L B_z$, g_L is the gyromagnetic ratio, and B_z is the magnetic field (along the z-axis), then Eq. (15.134) describes the Larmor precession of charged bodies in a uniform magnetic field B_z. ■

[14] D. Menzel, ed., *Fundamental Formulas of Physics*, Englewood Cliffs, NJ: Prentice-Hall (1955), reprinted, 2nd ed., Dover (1960), p. 695.

Dirac Delta Function

For use with differential equations one further transform is helpful — the Dirac delta function:[15]

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)\,dt = e^{-st_0}, \qquad \text{for } t_0 \ge 0, \tag{15.140}$$

and for $t_0 = 0$

$$\mathcal{L}\{\delta(t)\} = 1, \tag{15.141}$$

where it is assumed that we are using a representation of the delta function such that

$$\int_0^\infty \delta(t)\,dt = 1, \qquad \delta(t) = 0, \qquad \text{for } t > 0. \tag{15.142}$$

As an alternate method, $\delta(t)$ may be considered the limit as $\varepsilon \to 0$ of $F(t)$, where

$$F(t) = \begin{cases} 0, & t < 0, \\ \varepsilon^{-1}, & 0 < t < \varepsilon, \\ 0, & t > \varepsilon. \end{cases} \tag{15.143}$$

By direct calculation

$$\mathcal{L}\{F(t)\} = \frac{1 - e^{-\varepsilon s}}{\varepsilon s}. \tag{15.144}$$

Taking the limit of the integral (instead of the integral of the limit), we have

$$\lim_{\varepsilon \to 0} \mathcal{L}\{F(t)\} = 1,$$

or Eq. (15.141),

$$\mathcal{L}\{\delta(t)\} = 1.$$

This delta function is frequently called the **impulse** function because it is so useful in describing impulsive forces, that is, forces lasting only a short time.

Example 15.9.3 Impulsive Force

Newton's second law for impulsive force acting on a particle of mass m becomes

$$m\frac{d^2X}{dt^2} = P\delta(t), \tag{15.145}$$

where P is a constant. Transforming, we obtain

$$ms^2x(s) - msX(0) - mX'(0) = P. \tag{15.146}$$

[15] Strictly speaking, the Dirac delta function is undefined. However, the integral over it is well defined. This approach is developed in Section 1.16 using delta sequences.

For a particle starting from rest, $X'(0) = 0$.[16] We shall also take $X(0) = 0$. Then

$$x(s) = \frac{P}{ms^2}, \tag{15.147}$$

and

$$X(t) = \frac{P}{m}t, \tag{15.148}$$

$$\frac{dX(t)}{dt} = \frac{P}{m}, \qquad \text{a constant.} \tag{15.149}$$

The effect of the impulse $P\delta(t)$ is to transfer (instantaneously) P units of linear momentum to the particle.

A similar analysis applies to the ballistic galvanometer. The torque on the galvanometer is given initially by $k\iota$, in which ι is a pulse of current and k is a proportionality constant. Since ι is of short duration, we set

$$k\iota = kq\ \delta(t), \tag{15.150}$$

where q is the total charge carried by the current ι. Then, with I the moment of inertia,

$$I\frac{d^2\theta}{dt^2} = kq\ \delta(t), \tag{15.151}$$

and, transforming as before, we find that the effect of the current pulse is a transfer of kq units of **angular** momentum to the galvanometer. ■

Exercises

15.9.1 Use the expression for the transform of a second derivative to obtain the transform of $\cos kt$.

15.9.2 A mass m is attached to one end of an unstretched spring, spring constant k (Fig. 15.10). At time $t = 0$ the free end of the spring experiences a constant acceleration a, away from the mass. Using Laplace transforms,

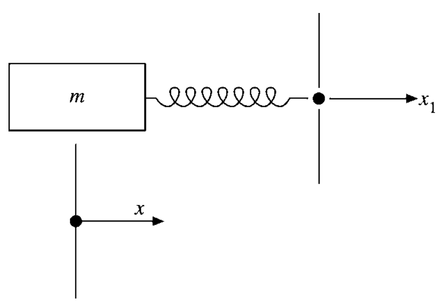

FIGURE 15.10 Spring.

[16]This should be $X'(+0)$. To include the effect of the impulse, consider that the impulse will occur at $t = \varepsilon$ and let $\varepsilon \to 0$.

(a) Find the position x of m as a function of time.
(b) Determine the limiting form of $x(t)$ for small t.

$$\text{ANS. (a) } x = \frac{1}{2}at^2 - \frac{a}{\omega^2}(1 - \cos\omega t), \quad \omega^2 = \frac{k}{m},$$
$$\text{(b) } x = \frac{a\omega^2}{4!}t^4, \quad \omega t \ll 1.$$

15.9.3 Radioactive nuclei decay according to the law

$$\frac{dN}{dt} = -\lambda N,$$

N being the concentration of a given nuclide and λ being the particular decay constant. This equation may be interpreted as stating that the rate of decay is proportional to the number of these radioactive nuclei present. They all decay independently.
In a radioactive series of n different nuclides, starting with N_1,

$$\frac{dN_1}{dt} = -\lambda_1 N_1,$$
$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2, \qquad \text{and so on.}$$
$$\frac{dN_n}{dt} = \lambda_{n-1} N_{n-1}, \qquad \text{stable.}$$

Find $N_1(t)$, $N_2(t)$, $N_3(t)$, $n = 3$, with $N_1(0) = N_0$, $N_2(0) = N_3(0) = 0$.

$$\text{ANS. } N_1(t) = N_0 e^{-\lambda_1 t}, \quad N_2(t) = N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right),$$
$$N_3(t) = N_0\left(1 - \frac{\lambda_2}{\lambda_2 - \lambda_1}e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1}e^{-\lambda_2 t}\right).$$

Find an approximate expression for N_2 and N_3, valid for small t when $\lambda_1 \approx \lambda_2$.

$$\text{ANS. } N_2 \approx N_0 \lambda_1 t, \quad N_3 \approx \frac{N_0}{2}\lambda_1 \lambda_2 t^2.$$

Find approximate expressions for N_2 and N_3, valid for large t, when

(a) $\lambda_1 \gg \lambda_2$,
(b) $\lambda_1 \ll \lambda_2$.

$$\text{ANS. (a) } N_2 \approx N_0 e^{-\lambda_2 t},$$
$$N_3 \approx N_0\left(1 - e^{-\lambda_2 t}\right), \quad \lambda_1 t \gg 1.$$
$$\text{(b) } N_2 \approx N_0 \tfrac{\lambda_1}{\lambda_2} e^{-\lambda_1 t},$$
$$N_3 \approx N_0\left(1 - e^{-\lambda_1 t}\right), \quad \lambda_2 t \gg 1.$$

15.9.4 The formation of an isotope in a nuclear reactor is given by

$$\frac{dN_2}{dt} = nv\sigma_1 N_{10} - \lambda_2 N_2(t) - nv\sigma_2 N_2(t).$$

Here the product nv is the neutron flux, neutrons per cubic centimeter, times centimeters per second mean velocity; σ_1 and σ_2 (cm^2) are measures of the probability of neutron absorption by the original isotope, concentration N_{10}, which is assumed constant and the newly formed isotope, concentration N_2, respectively. The radioactive decay constant for the isotope is λ_2.

(a) Find the concentration N_2 of the new isotope as a function of time.
(b) If the original element is Eu^{153}, $\sigma_1 = 400$ barns $= 400 \times 10^{-24}$ cm^2, $\sigma_2 = 1000$ barns $= 1000 \times 10^{-24}$ cm^2, and $\lambda_2 = 1.4 \times 10^{-9}\,\mathrm{s}^{-1}$. If $N_{10} = 10^{20}$ and $(nv) = 10^9\ \mathrm{cm}^{-2}\,\mathrm{s}^{-1}$, find N_2, the concentration of Eu^{154} after one year of continuous irradiation. Is the assumption that N_1 is constant justified?

15.9.5 In a nuclear reactor Xe^{135} is formed as both a direct fission product and a decay product of I^{135}, half-life, 6.7 hours. The half-life of Xe^{135} is 9.2 hours. Because Xe^{135} strongly absorbs thermal neutrons thereby “poisoning” the nuclear reactor, its concentration is a matter of great interest. The relevant equations are

$$\frac{dN_I}{dt} = \gamma_I \varphi \sigma_f N_U - \lambda_I N_I,$$
$$\frac{dN_X}{dt} = \lambda_I N_I + \gamma_X \varphi \sigma_f N_U - \lambda_X N_X - \varphi \sigma_X N_X.$$

Here $N_I =$ concentration of I^{135} (Xe^{135}, U^{235}). Assume

$$\begin{aligned}
N_U &= \text{constant},\\
\gamma_I &= \text{yield of } \mathrm{I}^{135} \text{ per fission} = 0.060,\\
\gamma_X &= \text{yield of } \mathrm{Xe}^{135} \text{ direct from fission} = 0.003,\\
\lambda_I &= \mathrm{I}^{135}\ \left(\mathrm{Xe}^{135}\right) \text{ decay constant} = \frac{\ln 2}{t_{1/2}} = \frac{0.693}{t_{1/2}},\\
\sigma_f &= \text{thermal neutron fission cross section for } \mathrm{U}^{235},\\
\sigma_X &= \text{thermal neutron absorption cross section for } \mathrm{Xe}^{135}\\
&= 3.5 \times 10^6 \text{ barns} = 3.5 \times 10^{-18}\ \mathrm{cm}^2.\\
&\quad (\sigma_I \text{ the absorption cross section of } \mathrm{I}^{135}\text{, is negligible.})\\
\varphi &= \text{neutron flux} = \text{neutrons/cm}^3 \times \text{mean velocity (cm/s)}.
\end{aligned}$$

(a) Find $N_X(t)$ in terms of neutron flux φ and the product $\sigma_f N_U$.
(b) Find $N_X(t \to \infty)$.
(c) After N_X has reached equilibrium, the reactor is shut down, $\varphi = 0$. Find $N_X(t)$ following shutdown. Notice the increase in N_X, which may for a few hours interfere with starting the reactor up again.

15.10 OTHER PROPERTIES

Substitution

If we replace the parameter s by $s - a$ in the definition of the Laplace transform (Eq. (15.99)), we have

$$\begin{aligned} f(s-a) &= \int_0^\infty e^{-(s-a)t} F(t)\,dt = \int_0^\infty e^{-st} e^{at} F(t)\,dt \\ &= \mathcal{L}\left\{e^{at} F(t)\right\}. \end{aligned} \tag{15.152}$$

Hence the replacement of s with $s - a$ corresponds to multiplying $F(t)$ by e^{at}, and conversely. This result can be used to good advantage in extending our table of transforms. From Eq. (15.107) we find immediately that

$$\mathcal{L}\left\{e^{at} \sin kt\right\} = \frac{k}{(s-a)^2 + k^2}; \tag{15.153}$$

also,

$$\mathcal{L}\left\{e^{at} \cos kt\right\} = \frac{s-a}{(s-a)^2 + k^2}, \qquad s > a.$$

Example 15.10.1 DAMPED OSCILLATOR

These expressions are useful when we consider an oscillating mass with damping proportional to the velocity. Equation (15.129), with such damping added, becomes

$$mX''(t) + bX'(t) + kX(t) = 0, \tag{15.154}$$

in which b is a proportionality constant. Let us assume that the particle starts from rest at $X(0) = X_0$, $X'(0) = 0$. The transformed equation is

$$m\left[s^2 x(s) - sX_0\right] + b\left[sx(s) - X_0\right] + kx(s) = 0, \tag{15.155}$$

and

$$x(s) = X_0 \frac{ms + b}{ms^2 + bs + k}. \tag{15.156}$$

This may be handled by completing the square of the denominator:

$$s^2 + \frac{b}{m}s + \frac{k}{m} = \left(s + \frac{b}{2m}\right)^2 + \left(\frac{k}{m} - \frac{b^2}{4m^2}\right). \tag{15.157}$$

If the damping is small, $b^2 < 4\,km$, the last term is positive and will be denoted by ω_1^2:

$$\begin{aligned} x(s) &= X_0 \frac{s + b/m}{(s + b/2m)^2 + \omega_1^2} \\ &= X_0 \frac{s + b/2m}{(s + b/2m)^2 + \omega_1^2} + X_0 \frac{(b/2m\omega_1)\omega_1}{(s + b/2m)^2 + \omega_1^2}. \end{aligned} \tag{15.158}$$

By Eq. (15.153),

$$X(t) = X_0 e^{-(b/2m)t}\left(\cos\omega_1 t + \frac{b}{2m\omega_1}\sin\omega_1 t\right)$$
$$= X_0 \frac{\omega_0}{\omega_1} e^{-(b/2m)t}\cos(\omega_1 t - \varphi), \tag{15.159}$$

where

$$\tan\varphi = \frac{b}{2m\omega_1}, \qquad \omega_0^2 = \frac{k}{m}.$$

Of course, as $b \to 0$, this solution goes over to the undamped solution (Section 15.9). ■

RLC Analog

It is worth noting the similarity between this damped simple harmonic oscillation of a mass on a spring and an *RLC* circuit (resistance, inductance, and capacitance) (Fig. 15.11). At any instant the sum of the potential differences around the loop must be zero (Kirchhoff's law, conservation of energy). This gives

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int^t I\,dt = 0. \tag{15.160}$$

Differentiating the current I with respect to time (to eliminate the integral), we have

$$L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0. \tag{15.161}$$

If we replace $I(t)$ with $X(t)$, L with m, R with b, and C^{-1} with k, then Eq. (15.161) is identical with the mechanical problem. It is but one example of the unification of diverse branches of physics by mathematics. A more complete discussion will be found in Olson's book.[17]

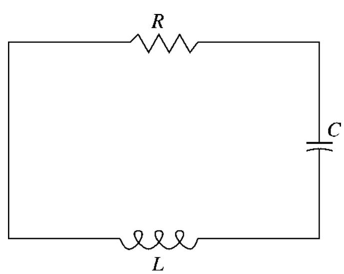

FIGURE 15.11 *RLC* circuit.

[17]H. F. Olson, *Dynamical Analogies*, New York: Van Nostrand (1943).

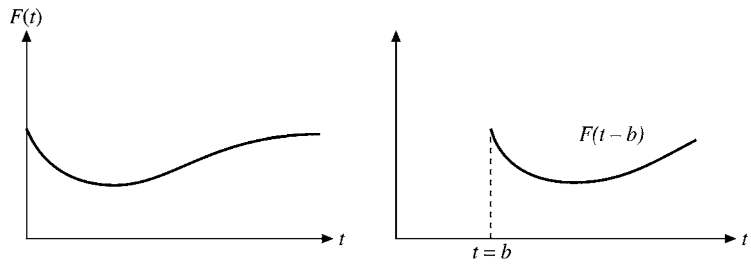

FIGURE 15.12 Translation.

Translation

This time let $f(s)$ be multiplied by e^{-bs}, $b > 0$:

$$e^{-bs} f(s) = e^{-bs} \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^\infty e^{-s(t+b)} F(t)\, dt. \tag{15.162}$$

Now let $t + b = \tau$. Equation (15.162) becomes

$$e^{-bs} f(s) = \int_b^\infty e^{-s\tau} F(\tau - b)\, d\tau$$

$$= \int_0^\infty e^{-s\tau} F(\tau - b) u(\tau - b)\, d\tau, \tag{15.163}$$

where $u(\tau - b)$ is the unit step function. This relation is often called the **Heaviside shifting theorem** (Fig. 15.12).

Since $F(t)$ is assumed to be equal to zero for $t < 0$, $F(\tau - b) = 0$ for $0 \le \tau < b$. Therefore we can extend the lower limit to zero without changing the value of the integral. Then, noting that τ is only a variable of integration, we obtain

$$e^{-bs} f(s) = \mathcal{L}\{F(t - b)\}. \tag{15.164}$$

Example 15.10.2 ELECTROMAGNETIC WAVES

The electromagnetic wave equation with $E = E_y$ or E_z, a transverse wave propagating along the x-axis, is

$$\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 E(x,t)}{\partial t^2} = 0. \tag{15.165}$$

Transforming this equation with respect to t, we get

$$\frac{\partial^2}{\partial x^2}\mathcal{L}\{E(x,t)\} - \frac{s^2}{v^2}\mathcal{L}\{E(x,t)\} + \frac{s}{v^2}E(x,0) + \frac{1}{v^2}\frac{\partial E(x,t)}{\partial t}\bigg|_{t=0} = 0. \tag{15.166}$$

If we have the initial condition $E(x, 0) = 0$ and

$$\left.\frac{\partial E(x,t)}{\partial t}\right|_{t=0} = 0,$$

then

$$\frac{\partial^2}{\partial x^2}\mathcal{L}\big\{E(x,t)\big\} = \frac{s^2}{v^2}\mathcal{L}\big\{E(x,t)\big\}. \tag{15.167}$$

The solution (of this **ODE**) is

$$\mathcal{L}\big\{E(x,t)\big\} = c_1 e^{-(s/v)x} + c_2 e^{+(s/v)x}. \tag{15.168}$$

The "constants" c_1 and c_2 are obtained by additional boundary conditions. They are constant with respect to x but may depend on s. If our wave remains finite as $x \to \infty$, $\mathcal{L}\{E(x,t)\}$ will also remain finite. Hence $c_2 = 0$. If $E(0,t)$ is denoted by $F(t)$, then $c_1 = f(s)$ and

$$\mathcal{L}\big\{E(x,t)\big\} = e^{-(s/v)x} f(s). \tag{15.169}$$

From the translation property (Eq. (15.164)) we find immediately that

$$E(x,t) = \begin{cases} F\big(t - \frac{x}{v}\big), & t \ge \frac{x}{v}, \\ 0, & t < \frac{x}{v}. \end{cases} \tag{15.170}$$

Differentiation and substitution into Eq. (15.165) verifies Eq. (15.170). Our solution represents a wave (or pulse) moving in the positive x-direction with velocity v. Note that for $x > vt$ the region remains undisturbed; the pulse has not had time to get there. If we had wanted a signal propagated along the negative x-axis, c_1 would have been set equal to 0 and we would have obtained

$$E(x,t) = \begin{cases} F\big(t + \frac{x}{v}\big), & t \ge -\frac{x}{v}, \\ 0, & t < -\frac{x}{v}, \end{cases} \tag{15.171}$$

a wave along the negative x-axis. ■

Derivative of a Transform

When $F(t)$, which is at least piecewise continuous, and s are chosen so that $e^{-st}F(t)$ converges exponentially for large s, the integral

$$\int_0^\infty e^{-st} F(t)\, dt$$

is uniformly convergent and may be differentiated (under the integral sign) with respect to s. Then

$$f'(s) = \int_0^\infty (-t)e^{-st} F(t)\, dt = \mathcal{L}\big\{-tF(t)\big\}. \tag{15.172}$$

Continuing this process, we obtain

$$f^{(n)}(s) = \mathcal{L}\big\{(-t)^n F(t)\big\}. \tag{15.173}$$

All the integrals so obtained will be uniformly convergent because of the decreasing exponential behavior of $e^{-st}F(t)$.

This same technique may be applied to generate more transforms. For example,

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st}e^{kt}\,dt = \frac{1}{s-k}, \qquad s > k. \tag{15.174}$$

Differentiating with respect to s (or with respect to k), we obtain

$$\mathcal{L}\{te^{kt}\} = \frac{1}{(s-k)^2}, \qquad s > k. \tag{15.175}$$

Example 15.10.3 BESSEL'S EQUATION

An interesting application of a differentiated Laplace transform appears in the solution of Bessel's equation with $n = 0$. From Chapter 11 we have

$$x^2 y''(x) + xy'(x) + x^2 y(x) = 0. \tag{15.176}$$

Dividing by x and substituting $t = x$ and $F(t) = y(x)$ to agree with the present notation, we see that the Bessel equation becomes

$$tF''(t) + F'(t) + tF(t) = 0. \tag{15.177}$$

We need a regular solution, in particular, $F(0) = 1$. From Eq. (15.177) with $t = 0$, $F'(+0) = 0$. Also, we assume that our unknown $F(t)$ has a transform. Transforming and using Eqs. (15.123), (15.124), and (15.172), we have

$$-\frac{d}{ds}\big[s^2 f(s) - s\big] + sf(s) - 1 - \frac{d}{ds}f(s) = 0. \tag{15.178}$$

Rearranging Eq. (15.178), we obtain

$$\big(s^2+1\big)f'(s) + sf(s) = 0, \tag{15.179}$$

or

$$\frac{df}{f} = -\frac{s\,ds}{s^2+1}, \tag{15.180}$$

a first-order ODE. By integration,

$$\ln f(s) = -\tfrac{1}{2}\ln\big(s^2+1\big) + \ln C, \tag{15.181}$$

which may be rewritten as

$$f(s) = \frac{C}{\sqrt{s^2+1}}. \tag{15.182}$$

To make use of Eq. (15.108), we expand $f(s)$ in a series of negative powers of s, convergent for $s > 1$:

$$\begin{aligned} f(s) &= \frac{C}{s}\left(1+\frac{1}{s^2}\right)^{-1/2} \\ &= \frac{C}{s}\left[1 - \frac{1}{2s^2} + \frac{1\cdot 3}{2^2\cdot 2!s^4} - \cdots + \frac{(-1)^n(2n)!}{(2^n n!)^2 s^{2n}} + \cdots\right]. \end{aligned} \tag{15.183}$$

Inverting, term by term, we obtain

$$F(t) = C\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2^n n!)^2}. \tag{15.184}$$

When C is set equal to 1, as required by the initial condition $F(0) = 1$, $F(t)$ is just $J_0(t)$, our familiar Bessel function of order zero. Hence

$$\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}. \tag{15.185}$$

Note that we assumed $s > 1$. The proof for $s > 0$ is left as a problem.

It is worth noting that this application was successful and relatively easy because we took $n = 0$ in Bessel's equation. This made it possible to divide out a factor of x (or t). If this had not been done, the terms of the form $t^2F(t)$ would have introduced a second derivative of $f(s)$. The resulting equation would have been no easier to solve than the original one.

When we go beyond linear ODEs with constant coefficients, the Laplace transform may still be applied, but there is no guarantee that it will be helpful.

The application to Bessel's equation, $n \neq 0$, will be found in the references. Alternatively, we can show that

$$\mathcal{L}\{J_n(at)\} = \frac{a^{-n}(\sqrt{s^2+a^2}-s)^n}{\sqrt{s^2+a^2}} \tag{15.186}$$

by expressing $J_n(t)$ as an infinite series and transforming term by term. ■

Integration of Transforms

Again, with $F(t)$ at least piecewise continuous and x large enough so that $e^{-xt}F(t)$ decreases exponentially (as $x \to \infty$), the integral

$$f(x) = \int_0^{\infty} e^{-xt}F(t)\,dt \tag{15.187}$$

is uniformly convergent with respect to x. This justifies reversing the order of integration in the following equation:

$$\begin{aligned}\int_s^b f(x)\,dx &= \int_s^b dx\int_0^{\infty} dt\,e^{-xt}F(t)\\ &= \int_0^{\infty}\frac{F(t)}{t}\left(e^{-st}-e^{-bt}\right)dt,\end{aligned} \tag{15.188}$$

on integrating with respect to x. The lower limit s is chosen large enough so that $f(s)$ is within the region of uniform convergence. Now letting $b \to \infty$, we have

$$\int_s^{\infty} f(x)\,dx = \int_0^{\infty}\frac{F(t)}{t}e^{-st}dt = \mathcal{L}\left\{\frac{F(t)}{t}\right\}, \tag{15.189}$$

provided that $F(t)/t$ is finite at $t = 0$ or diverges less strongly than t^{-1} (so that $\mathcal{L}\{F(t)/t\}$ will exist).

Limits of Integration — Unit Step Function

The actual limits of integration for the Laplace transform may be specified with the (Heaviside) unit step function

$$u(t-k) = \begin{cases} 0, & t < k \\ 1, & t > k. \end{cases}$$

For instance,

$$\mathcal{L}\{u(t-k)\} = \int_k^\infty e^{-st}\,dt = \frac{1}{s}e^{-ks}.$$

A rectangular pulse of width k and unit height is described by $F(t) = u(t) - u(t-k)$. Taking the Laplace transform, we obtain

$$\mathcal{L}\{u(t) - u(t-k)\} = \int_0^k e^{-st}\,dt = \frac{1}{s}\left(1 - e^{-ks}\right).$$

The unit step function is also used in Eq. (15.163) and could be invoked in Exercise 15.10.13.

Exercises

15.10.1 Solve Eq. (15.154), which describes a damped simple harmonic oscillator for $X(0) = X_0$, $X'(0) = 0$, and

(a) $b^2 = 4$ km (critically damped),
(b) $b^2 > 4$ km (overdamped).

ANS. (a) $X(t) = X_0 e^{-(b/2m)t}\left(1 + \dfrac{b}{2m}t\right)$.

15.10.2 Solve Eq. (15.154), which describes a damped simple harmonic oscillator for $X(0) = 0$, $X'(0) = v_0$, and

(a) $b^2 < 4$ km (underdamped),
(b) $b^2 = 4$ km (critically damped),
(c) $b^2 > 4$ km (overdamped).

ANS. (a) $X(t) = \dfrac{v_0}{\omega_1} e^{-(b/2m)t} \sin\omega_1 t$,
(b) $X(t) = v_0 t e^{-(b/2m)t}$.

15.10.3 The motion of a body falling in a resisting medium may be described by

$$m\frac{d^2X(t)}{dt^2} = mg - b\frac{dX(t)}{dt}$$

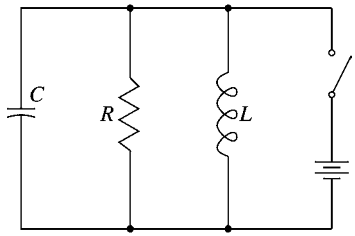

FIGURE 15.13 Ringing circuit.

when the retarding force is proportional to the velocity. Find $X(t)$ and $dX(t)/dt$ for the initial conditions

$$X(0) = \left.\frac{dX}{dt}\right|_{t=0} = 0.$$

15.10.4 **Ringing circuit**. In certain electronic circuits, resistance, inductance, and capacitance are placed in the plate circuit in parallel (Fig. 15.13). A constant voltage is maintained across the parallel elements, keeping the capacitor charged. At time $t = 0$ the circuit is disconnected from the voltage source. Find the voltages across the parallel elements R, L, and C as a function of time. Assume R to be large.
Hint. By Kirchhoff's laws

$$I_R + I_C + I_L = 0 \qquad \text{and} \qquad E_R = E_C = E_L,$$

where

$$E_R = I_R R, \qquad E_L = L\frac{dI_L}{dt}$$

$$E_C = \frac{q_0}{C} + \frac{1}{C}\int_0^t I_C\,dt,$$

$$q_0 = \text{initial charge of capacitor.}$$

With the DC impedance of $L = 0$, let $I_L(0) = I_0$, $E_L(0) = 0$. This means $q_0 = 0$.

15.10.5 With $J_0(t)$ expressed as a contour integral, apply the Laplace transform operation, reverse the order of integration, and thus show that

$$\mathcal{L}\big\{J_0(t)\big\} = \big(s^2+1\big)^{-1/2}, \qquad \text{for } s > 0.$$

15.10.6 Develop the Laplace transform of $J_n(t)$ from $\mathcal{L}\{J_0(t)\}$ by using the Bessel function recurrence relations.
Hint. Here is a chance to use mathematical induction.

15.10.7 A calculation of the magnetic field of a circular current loop in circular cylindrical coordinates leads to the integral

$$\int_0^\infty e^{-kz} k J_1(ka)\,dk, \qquad \Re(z) \geq 0.$$

Show that this integral is equal to $a/(z^2+a^2)^{3/2}$.

15.10.8 The electrostatic potential of a point charge q at the origin in circular cylindrical coordinates is

$$\frac{q}{4\pi\varepsilon_0}\int_0^\infty e^{-kz}J_0(k\rho)\,dk = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{(\rho^2+z^2)^{1/2}}, \qquad \Re(z)\geq 0.$$

From this relation show that the Fourier cosine and sine transforms of $J_0(k\rho)$ are

(a) $$\sqrt{\frac{\pi}{2}}F_c\{J_0(k\rho)\} = \int_0^\infty J_0(k\rho)\cos k\zeta\,dk = \begin{cases}\left(\rho^2-\zeta^2\right)^{-1/2}, & \rho>\zeta,\\ 0, & \rho<\zeta.\end{cases}$$

(b) $$\sqrt{\frac{\pi}{2}}F_s\{J_0(k\rho)\} = \int_0^\infty J_0(k\rho)\sin k\zeta\,dk = \begin{cases}0, & \rho>\zeta,\\ \left(\rho^2-\zeta^2\right)^{-1/2}, & \rho<\zeta.\end{cases}$$

Hint. Replace z by $z+i\zeta$ and take the limit as $z\to 0$.

15.10.9 Show that

$$\mathcal{L}\{I_0(at)\} = \left(s^2-a^2\right)^{-1/2}, \qquad s>a.$$

15.10.10 Verify the following Laplace transforms:

(a) $\mathcal{L}\{j_0(at)\} = \mathcal{L}\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\cot^{-1}\left(\frac{s}{a}\right)$,

(b) $\mathcal{L}\{n_0(at)\}$ does not exist,

(c) $\mathcal{L}\{i_0(at)\} = \mathcal{L}\left\{\frac{\sinh at}{at}\right\} = \frac{1}{2a}\ln\frac{s+a}{s-a} = \frac{1}{a}\coth^{-1}\left(\frac{s}{a}\right)$,

(d) $\mathcal{L}\{k_0(at)\}$ does not exist.

15.10.11 Develop a Laplace transform solution of Laguerre's equation

$$tF''(t)+(1-t)F'(t)+nF(t)=0.$$

Note that you need a derivative of a transform and a transform of derivatives. Go as far as you can with n; then (and only then) set $n=0$.

15.10.12 Show that the Laplace transform of the Laguerre polynomial $L_n(at)$ is given by

$$\mathcal{L}\{L_n(at)\} = \frac{(s-a)^n}{s^{n+1}}, \qquad s>0.$$

15.10.13 Show that

$$\mathcal{L}\{E_1(t)\} = \frac{1}{s}\ln(s+1), \qquad s>0,$$

where

$$E_1(t) = \int_t^\infty \frac{e^{-\tau}}{\tau}\,d\tau = \int_1^\infty \frac{e^{-xt}}{x}dx.$$

$E_1(t)$ is the exponential-integral function.

15.10.14 (a) From Eq. (15.189) show that

$$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{F(t)}{t}dt,$$

provided the integrals exist.

(b) From the preceding result show that

$$\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2},$$

in agreement with Eqs. (15.122) and (7.56).

15.10.15 (a) Show that

$$\mathcal{L}\left\{\frac{\sin kt}{t}\right\} = \cot^{-1}\left(\frac{s}{k}\right).$$

(b) Using this result (with $k = 1$), prove that

$$\mathcal{L}\{\mathrm{si}(t)\} = -\frac{1}{s}\tan^{-1} s,$$

where

$$\mathrm{si}(t) = -\int_t^\infty \frac{\sin x}{x}dx, \qquad \text{the sine integral.}$$

15.10.16 If $F(t)$ is periodic (Fig. 15.14) with a period a so that $F(t+a) = F(t)$ for all $t \geq 0$, show that

$$\mathcal{L}\{F(t)\} = \frac{\int_0^a e^{-st}F(t)\,dt}{1 - e^{-as}},$$

with the integration now over only the **first period** of $F(t)$.

15.10.17 Find the Laplace transform of the square wave (period a) defined by

$$F(t) = \begin{cases} 1, & 0 < t < \frac{a}{2} \\ 0, & \frac{a}{2} < t < a. \end{cases}$$

ANS. $f(s) = \dfrac{1}{s} \cdot \dfrac{1 - e^{-as/2}}{1 - e^{-as}}.$

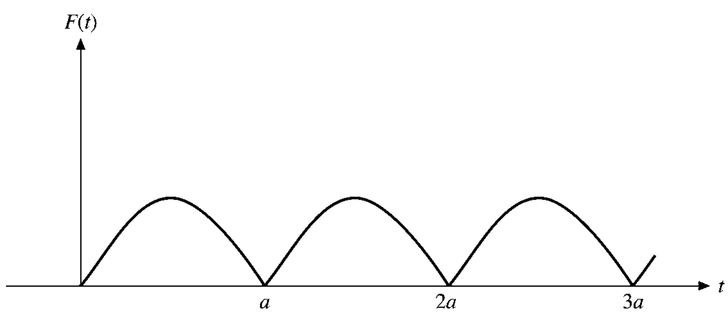

FIGURE 15.14 Periodic function.

15.10.18 Show that

(a) $\mathcal{L}\{\cosh at \cos at\} = \dfrac{s^3}{s^4 + 4a^4}$,

(b) $\mathcal{L}\{\cosh at \sin at\} = \dfrac{as^2 + 2a^3}{s^4 + 4a^4}$,

(c) $\mathcal{L}\{\sinh at \cos at\} = \dfrac{as^2 - 2a^3}{s^4 + 4a^4}$,

(d) $\mathcal{L}\{\sinh at \sin at\} = \dfrac{2a^2 s}{s^4 + 4a^4}$.

15.10.19 Show that

(a) $\mathcal{L}^{-1}\left\{\left(s^2 + a^2\right)^{-2}\right\} = \dfrac{1}{2a^3} \sin at - \dfrac{1}{2a^2} t \cos at$,

(b) $\mathcal{L}^{-1}\left\{s\left(s^2 + a^2\right)^{-2}\right\} = \dfrac{1}{2a} t \sin at$,

(c) $\mathcal{L}^{-1}\left\{s^2\left(s^2 + a^2\right)^{-2}\right\} = \dfrac{1}{2a} \sin at + \dfrac{1}{2} t \cos at$,

(d) $\mathcal{L}^{-1}\left\{s^3\left(s^2 + a^2\right)^{-2}\right\} = \cos at - \dfrac{a}{2} t \sin at$.

15.10.20 Show that

$$\mathcal{L}\left\{\left(t^2 - k^2\right)^{-1/2} u(t - k)\right\} = K_0(ks).$$

Hint. Try transforming an integral representation of $K_0(ks)$ into the Laplace transform integral.

15.10.21 The Laplace transform

$$\int_0^\infty e^{-xs} x J_0(x)\, dx = \frac{s}{(s^2 + 1)^{3/2}}$$

may be rewritten as

$$\frac{1}{s^2} \int_0^\infty e^{-y} y J_0\left(\frac{y}{s}\right) dy = \frac{s}{(s^2 + 1)^{3/2}},$$

which is in Gauss–Laguerre quadrature form. Evaluate this integral for $s = 1.0, 0.9, 0.8, \ldots$, decreasing s in steps of 0.1 until the relative error rises to 10 percent. (The effect of decreasing s is to make the integrand oscillate more rapidly per unit length of y, thus decreasing the accuracy of the numerical quadrature.)

15.10.22 (a) Evaluate

$$\int_0^\infty e^{-kz} k J_1(ka)\, dk$$

by the Gauss–Laguerre quadrature. Take $a = 1$ and $z = 0.1(0.1)1.0$.

(b) From the analytic form, Exercise 15.10.7, calculate the absolute error and the relative error.

15.11 CONVOLUTION (FALTUNGS) THEOREM

One of the most important properties of the Laplace transform is that given by the convolution, or Faltungs, theorem.[18] We take two transforms,

$$f_1(s)=\mathcal{L}\{F_1(t)\} \qquad \text{and} \qquad f_2(s)=\mathcal{L}\{F_2(t)\}, \tag{15.190}$$

and multiply them together. To avoid complications when changing variables, we hold the upper limits finite:

$$f_1(s)f_2(s)=\lim_{a\to\infty}\int_0^a e^{-sx}F_1(x)\,dx\int_0^{a-x}e^{-sy}F_2(y)\,dy. \tag{15.191}$$

The upper limits are chosen so that the area of integration, shown in Fig. 15.15a, is the shaded triangle, not the square. If we integrate over a square in the xy-plane, we have a parallelogram in the tz-plane, which simply adds complications. This modification is permissible because the two integrands are assumed to decrease exponentially. In the limit $a\to\infty$, the integral over the unshaded triangle will give zero contribution. Substituting $x=t-z$, $y=z$, the region of integration is mapped into the triangle shown in Fig. 15.15b. To verify the mapping, map the vertices: $t=x+y$, $z=y$. Using Jacobians to transform the element of area, we have

$$dx\,dy=\begin{vmatrix}\dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t}\\[2mm] \dfrac{\partial x}{\partial z} & \dfrac{\partial y}{\partial z}\end{vmatrix}dt\,dz=\begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix}dt\,dz \tag{15.192}$$

or $dx\,dy=dt\,dz$. With this substitution Eq. (15.191) becomes

$$\begin{aligned}f_1(s)f_2(s)&=\lim_{a\to\infty}\int_0^a e^{-st}\int_0^t F_1(t-z)F_2(z)\,dz\,dt\\ &=\mathcal{L}\left\{\int_0^t F_1(t-z)F_2(z)\,dz\right\}.\end{aligned} \tag{15.193}$$

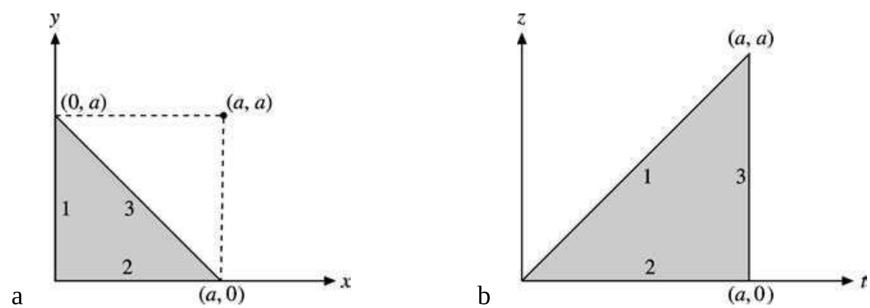

FIGURE 15.15
Change of variables,
(a) xy-plane (b) zt-plane.

[18]An alternate derivation employs the Bromwich integral (Section 15.12). This is Exercise 15.12.3.

For convenience this integral is represented by the symbol

$$\int_0^t F_1(t-z)F_2(z)\,dz \equiv F_1 * F_2 \tag{15.194}$$

and referred to as the **convolution**, closely analogous to the Fourier convolution (Section 15.5). If we substitute $w = t - z$, we find

$$F_1 * F_2 = F_2 * F_1, \tag{15.195}$$

showing that the relation is symmetric.

Carrying out the inverse transform, we also find

$$\mathcal{L}^{-1}\big\{f_1(s)f_2(s)\big\} = \int_0^t F_1(t-z)F_2(z)\,dz. \tag{15.196}$$

This can be useful in the development of new transforms or as an alternative to a partial fraction expansion. One immediate application is in the solution of integral equations (Section 16.2). Since the upper limit, t, is variable, this Laplace convolution is useful in treating Volterra integral equations. The Fourier convolution with fixed (infinite) limits would apply to Fredholm integral equations.

Example 15.11.1 Driven Oscillator with Damping

As one illustration of the use of the convolution theorem, let us return to the mass m on a spring, with damping and a driving force $F(t)$. The equation of motion ((15.129) or (15.154)) now becomes

$$mX''(t) + bX'(t) + kX(t) = F(t). \tag{15.197}$$

Initial conditions $X(0) = 0$, $X'(0) = 0$ are used to simplify this illustration, and the transformed equation is

$$ms^2x(s) + bs\,x(s) + kx(s) = f(s), \tag{15.198}$$

or

$$x(s) = \frac{f(s)}{m}\frac{1}{(s+b/2m)^2+\omega_1^2}, \tag{15.199}$$

where $\omega_1^2 \equiv k/m - b^2/4m^2$, as before.

By the convolution theorem (Eq. (15.193) or (15.196)),

$$X(t) = \frac{1}{m\omega_1}\int_0^t F(t-z)e^{-(b/2m)z}\sin\omega_1 z\,dz. \tag{15.200}$$

If the force is impulsive, $F(t) = P\delta(t)$,[19]

$$X(t) = \frac{P}{m\omega_1}e^{-(b/2m)t}\sin\omega_1 t. \tag{15.201}$$

[19]Note that $\delta(t)$ lies **inside** the interval $[0, t]$.

P represents the momentum transferred by the impulse, and the constant P/m takes the place of an initial velocity $X'(0)$.

If $F(t) = F_0 \sin \omega t$, Eq. (15.200) may be used, but a partial fraction expansion is perhaps more convenient. With

$$f(s) = \frac{F_0 \omega}{s^2 + \omega^2}$$

Eq. (15.199) becomes

$$\begin{aligned} x(s) &= \frac{F_0 \omega}{m} \cdot \frac{1}{s^2 + \omega^2} \cdot \frac{1}{(s + b/2m)^2 + \omega_1^2} \\ &= \frac{F_0 \omega}{m} \left[\frac{a's + b'}{s^2 + \omega^2} + \frac{c's + d'}{(s + b/2m)^2 + \omega_1^2} \right]. \end{aligned} \tag{15.202}$$

The coefficients a', b', c', and d' are independent of s. Direct calculation shows

$$\begin{aligned} -\frac{1}{a'} &= \frac{b}{m}\omega^2 + \frac{m}{b}\left(\omega_0^2 - \omega^2\right)^2, \\ -\frac{1}{b'} &= -\frac{m}{b}\left(\omega_0^2 - \omega^2\right)\left[\frac{b}{m}\omega^2 + \frac{m}{b}\left(\omega_0^2 - \omega^2\right)^2\right]. \end{aligned}$$

Since c' and d' will lead to exponentially decreasing terms (transients), they will be discarded here. Carrying out the inverse operation, we find for the steady-state solution

$$X(t) = \frac{F_0}{[b^2\omega^2 + m^2(\omega_0^2 - \omega^2)^2]^{1/2}} \sin(\omega t - \varphi), \tag{15.203}$$

where

$$\tan \varphi = \frac{b\omega}{m(\omega_0^2 - \omega^2)}.$$

Differentiating the denominator, we find that the amplitude has a maximum when

$$\omega^2 = \omega_0^2 - \frac{b^2}{2m^2} = \omega_1^2 - \frac{b^2}{4m^2}. \tag{15.204}$$

This is the resonance condition.[20] At resonance the amplitude becomes $F_0/b\omega_1$, showing that the mass m goes into infinite oscillation at resonance if damping is neglected $(b = 0)$. It is worth noting that we have had three different characteristic frequencies:

$$\omega_2^2 = \omega_0^2 - \frac{b^2}{2m^2},$$

resonance for forced oscillations, with damping;

$$\omega_1^2 = \omega_0^2 - \frac{b^2}{4m^2},$$

[20] The amplitude (squared) has the typical resonance denominator, the Lorentz line shape, Exercise 15.3.9.

free oscillation frequency, with damping; and

$$\omega_0^2 = \frac{k}{m},$$

free oscillation frequency, no damping. They coincide only if the damping is zero. ∎

Returning to Eqs. (15.197) and (15.199), Eq. (15.197) is our ODE for the response of a dynamical system to an arbitrary driving force. The final response clearly depends on both the driving force and the characteristics of our system. This dual dependence is separated in the transform space. In Eq. (15.199) the transform of the response (output) appears as the product of two factors, one describing the driving force (input) and the other describing the dynamical system. This latter part, which modifies the input and yields the output, is often called a **transfer function**. Specifically, $[(s+b/2m)^2+\omega_1^2]^{-1}$ is the transfer function corresponding to this damped oscillator. The concept of a transfer function is of great use in the field of servomechanisms. Often the characteristics of a particular servomechanism are described by giving its transfer function. The convolution theorem then yields the output signal for a particular input signal.

Exercises

15.11.1 From the convolution theorem show that

$$\frac{1}{s} f(s) = \mathcal{L}\left\{\int_0^t F(x)\,dx\right\},$$

where $f(s) = \mathcal{L}\{F(t)\}$.

15.11.2 If $F(t) = t^a$ and $G(t) = t^b$, $a > -1$, $b > -1$:

(a) Show that the convolution

$$F * G = t^{a+b+1} \int_0^1 y^a (1-y)^b \, dy.$$

(b) By using the convolution theorem, show that

$$\int_0^1 y^a (1-y)^b \, dy = \frac{a!b!}{(a+b+1)!}.$$

This is the Euler formula for the beta function (Eq. (8.59a)).

15.11.3 Using the convolution integral, calculate

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\}, \qquad a^2 \neq b^2.$$

15.11.4 An undamped oscillator is driven by a force $F_0 \sin\omega t$. Find the displacement as a function of time. Notice that it is a linear combination of two simple harmonic motions, one with the frequency of the driving force and one with the frequency ω_0 of the free oscillator. (Assume $X(0) = X'(0) = 0$.)

ANS. $X(t) = \dfrac{F_0/m}{\omega^2 - \omega_0^2}\left(\dfrac{\omega}{\omega_0}\sin\omega_0 t - \sin\omega t\right).$

Other exercises involving the Laplace convolution appear in Section 16.2.

15.12 INVERSE LAPLACE TRANSFORM

Bromwich Integral

We now develop an expression for the inverse Laplace transform $\mathcal{L}^{-1}$ appearing in the equation

$$F(t) = \mathcal{L}^{-1}\{f(s)\}. \tag{15.205}$$

One approach lies in the Fourier transform, for which we know the inverse relation. There is a difficulty, however. Our Fourier transformable function had to satisfy the Dirichlet conditions. In particular, we required that

$$\lim_{\omega\to\infty} G(\omega) = 0 \tag{15.206}$$

so that the infinite integral would be well defined.[21] Now we wish to treat functions $F(t)$ that may diverge exponentially. To surmount this difficulty, we extract an exponential factor, $e^{\gamma t}$, from our (possibly) divergent Laplace function and write

$$F(t) = e^{\gamma t} G(t). \tag{15.207}$$

If $F(t)$ diverges as $e^{\alpha t}$, we require γ to be greater than α so **that $G(t)$ will be convergent**. Now, with $G(t) = 0$ for $t < 0$ and otherwise suitably restricted so that it may be represented by a Fourier integral (Eq. (15.20)),

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iut}\,du \int_0^{\infty} G(v)e^{-iuv}\,dv. \tag{15.208}$$

Using Eq. (15.207), we may rewrite (15.208) as

$$F(t) = \frac{e^{\gamma t}}{2\pi}\int_{-\infty}^{\infty} e^{iut}\,du \int_0^{\infty} F(v)e^{-\gamma v}e^{-iuv}\,dv. \tag{15.209}$$

Now, with the change of variable,

$$s = \gamma + iu, \tag{15.210}$$

the integral over v is thrown into the form of a Laplace transform,

$$\int_0^{\infty} f(v)e^{-sv}\,dv = f(s); \tag{15.211}$$

[21] If delta functions are included, $G(\omega)$ may be a cosine. Although this does not satisfy Eq. (15.206), $G(\omega)$ is still bounded.

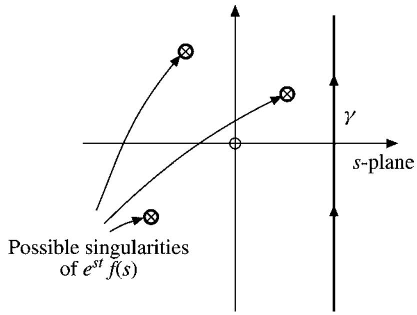

FIGURE 15.16 Singularities of $e^{st} f(s)$.

s is now a complex variable, and $\Re(s) \geq \gamma$ to guarantee convergence. Notice that the Laplace transform has mapped a function specified on the positive real axis onto the complex plane, $\Re(s) \geq \gamma$.[22]

With γ as a constant, $ds = i\,du$. Substituting Eq. (15.211) into Eq. (15.209), we obtain

$$F(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} f(s)\, ds. \tag{15.212}$$

Here is our **inverse transform**. We have rotated the line of integration through 90° (by using $ds = i\,du$). The path has become an infinite vertical line in the complex plane, the constant γ having been chosen so that all the singularities of $f(s)$ are on the left-hand side (Fig. 15.16).

Equation (15.212), our inverse transformation, is usually known as the **Bromwich integral**, although sometimes it is referred to as the **Fourier–Mellin theorem** or **Fourier–Mellin integral**. This integral may now be evaluated by the regular methods of contour integration (Chapter 7). If $t > 0$, the contour may be closed by an infinite semicircle in the left half-plane. Then by the residue theorem (Section 7.1)

$$F(t) = \Sigma(\text{residues included for } \Re(s) < \gamma). \tag{15.213}$$

Possibly this means of evaluation with $\Re(s)$ ranging through negative values seems paradoxical in view of our previous requirement that $\Re(s) \geq \gamma$. The paradox disappears when we recall that the requirement $\Re(s) \geq \gamma$ was imposed to guarantee convergence of the Laplace transform integral that defined $f(s)$. Once $f(s)$ is obtained, we may then proceed to exploit its properties as an analytical function in the complex plane wherever we choose.[23] In effect we are employing analytic continuation to get $\mathcal{L}\{F(t)\}$ in the left half-plane, exactly as the recurrence relation for the factorial function was used to extend the Euler integral definition (Eq. (8.5)) to the left half-plane.

Perhaps a pair of examples may clarify the evaluation of Eq. (15.212).

[22]For a derivation of the inverse Laplace transform using only real variables, see C. L. Bohn and R. W. Flynn, Real variable inversion of Laplace transforms: An application in plasma physics. *Am. J. Phys.* **46**: 1250 (1978).

[23]In numerical work $f(s)$ may well be available only for discrete real, positive values of s. Then numerical procedures are indicated. See Krylov and Skoblya in the Additional Reading.

Example 15.12.1 Inversion via Calculus of Residues

If $f(s) = a/(s^2 - a^2)$, then

$$e^{st} f(s) = \frac{ae^{st}}{s^2 - a^2} = \frac{ae^{st}}{(s+a)(s-a)}. \tag{15.214}$$

The residues may be found by using Exercise 6.6.1 or various other means. The first step is to identify the singularities, the poles. Here we have one simple pole at $s = a$ and another simple pole at $s = -a$. By Exercise 6.6.1, the residue at $s = a$ is $(\frac{1}{2})e^{at}$ and the residue at $s = -a$ is $(-\frac{1}{2})e^{-at}$. Then

$$\text{Residues} = \left(\tfrac{1}{2}\right)\left(e^{at} - e^{-at}\right) = \sinh at = F(t), \tag{15.215}$$

in agreement with Eq. (15.105). ■

Example 15.12.2

If

$$f(s) = \frac{1 - e^{-as}}{s},$$

then $e^{s(t-a)}$ grows exponentially for $t < a$ on the semicircle in the left-hand s-plane, so contour integration and the residue theorem are not applicable. However, we can evaluate the integral explicitly as follows. We let $\gamma \to 0$ and substitute $s = iy$, so

$$F(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} f(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[e^{iyt} - e^{iy(t-a)}\right] \frac{dy}{y}. \tag{15.216}$$

Using the Euler identity, only the sines survive that are odd in y and we obtain

$$F(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[\frac{\sin ty}{y} - \frac{\sin(t-a)y}{y}\right]. \tag{15.217}$$

If $k > 0$, then $\int_0^\infty \frac{\sin ky}{y} dy$ gives $\pi/2$, and it gives $-\pi/2$ if $k < 0$. As a consequence, $F(t) = 0$ if $t > a > 0$ and if $t < 0$. If $0 < t < a$, then $F(t) = 1$. This can be written compactly in terms of the Heaviside unit step function $u(t)$ as follows:

$$F(t) = u(t) - u(t-a) = \begin{cases} 0, & t < 0, \\ 1, & 0 < t < a, \\ 0, & t > a, \end{cases} \tag{15.218}$$

a step function of unit height and length a (Fig. 15.17). ■

Two general comments may be in order. First, these two examples hardly begin to show the usefulness and power of the Bromwich integral. It is always available for inverting a complicated transform when the tables prove inadequate.

Second, this derivation is not presented as a rigorous one. Rather, it is given more as a plausibility argument, although it can be made rigorous. The determination of the inverse transform is somewhat similar to the solution of a differential equation. It makes

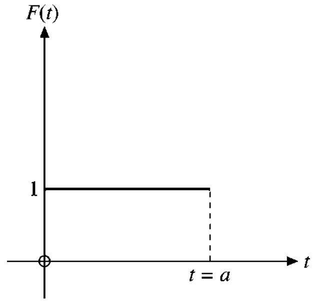

FIGURE 15.17
Finite-length step function
$u(t) - u(t-a)$.

little difference how you get the solution. Guess at it if you want. The solution can always be checked by substitution back into the original differential equation. Similarly, $F(t)$ can (and, to check on careless errors, should) be checked by determining whether, by Eq. (15.99),

$$\mathcal{L}\{F(t)\} = f(s).$$

Two alternate derivations of the Bromwich integral are the subjects of Exercises 15.12.1 and 15.12.2.

As a final illustration of the use of the Laplace inverse transform, we have some results from the work of Brillouin and Sommerfeld (1914) in electromagnetic theory.

Example 15.12.3 VELOCITY OF ELECTROMAGNETIC WAVES IN A DISPERSIVE MEDIUM

The group velocity u of traveling waves is related to the phase velocity v by the equation

$$u = v - \lambda \frac{dv}{d\lambda}. \tag{15.219}$$

Here λ is the wavelength. In the vicinity of an absorption line (resonance), $dv/d\lambda$ may be sufficiently negative so that $u > c$ (Fig. 15.18). The question immediately arises whether a signal can be transmitted faster than c, the velocity of light in vacuum. This question, which assumes that such a group velocity is meaningful, is of fundamental importance to the theory of special relativity.

We need a solution to the wave equation

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}, \tag{15.220}$$

corresponding to a harmonic vibration starting at the origin at time zero. Since our medium is dispersive, v is a function of the angular frequency. Imagine, for instance, a plane wave, angular frequency ω, incident on a shutter at the origin. At $t = 0$ the shutter is (instantaneously) opened, and the wave is permitted to advance along the positive x-axis.

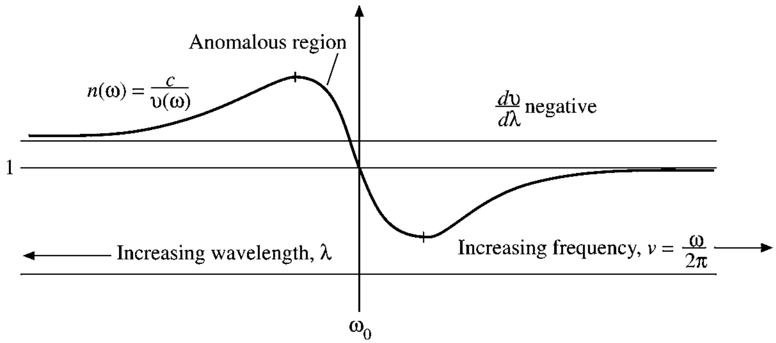

FIGURE 15.18 Optical dispersion.

Let us then build up a solution starting at $x = 0$. It is convenient to use the Cauchy integral formula, Eq. (6.43),

$$\psi(0,t) = \frac{1}{2\pi i}\oint \frac{e^{-izt}}{z - z_0}dz = e^{-iz_0 t}$$

(for a contour encircling $z = z_0$ in the positive sense). Using $s = -iz$ and $z_0 = \omega$, we obtain

$$\psi(0,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{s + i\omega}ds = \begin{cases} 0, & t < 0, \\ e^{-i\omega t}, & t > 0. \end{cases} \tag{15.221}$$

To be complete, the loop integral is along the vertical line $\Re(s) = \gamma$ **and** an infinite semicircle, as shown in Fig. 15.19. The location of the infinite semicircle is chosen so that the integral over it vanishes. This means a semicircle in the left half-plane for $t > 0$ and the residue is enclosed. For $t < 0$ we pick the right half-plane and no singularity is enclosed. The fact that this is just the Bromwich integral may be verified by noting that

$$F(t) = \begin{cases} 0, & t < 0, \\ e^{-i\omega t}, & t > 0 \end{cases} \tag{15.222}$$

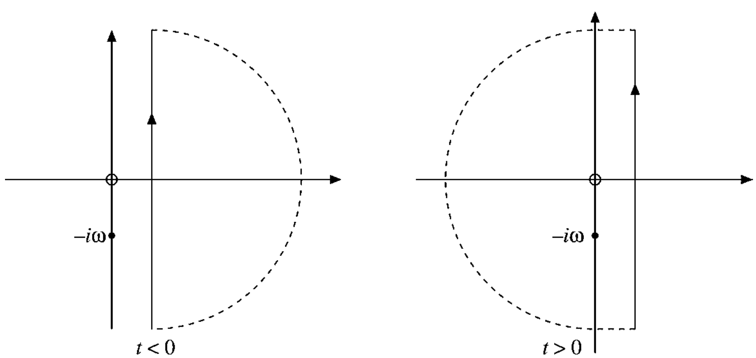

FIGURE 15.19 Possible closed contours.

and applying the Laplace transform. The transformed function $f(s)$ becomes

$$f(s) = \frac{1}{s + i\omega}. \tag{15.223}$$

Our Cauchy–Bromwich integral provides us with the time dependence of a signal leaving the origin at $t = 0$. To include the space dependence, we note that

$$e^{s(t-x/v)}$$

satisfies the wave equation. With this as a clue, we replace t by $t - x/v$ and write a solution:

$$\psi(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{s(t-x/v)}}{s+i\omega}\,ds. \tag{15.224}$$

It was seen in the derivation of the Bromwich integral that our variable s replaces the ω of the Fourier transformation. Hence the wave velocity v may become a function of s, that is, $v(s)$. Its particular form need not concern us here. We need only the property $v \le c$ and

$$\lim_{|s|\to\infty} v(s) = \text{constant},\ c. \tag{15.225}$$

This is suggested by the asymptotic behavior of the curve on the right side of Fig. 15.18.[24]

Evaluating Eq. (15.225) by the calculus of residues, we may close the path of integration by a semicircle in the right half-plane, provided

$$t - \frac{x}{c} < 0.$$

Hence

$$\psi(x,t) = 0, \qquad t - \frac{x}{c} < 0, \tag{15.226}$$

which means that the velocity of our signal cannot exceed the velocity of light in the vacuum, c. This simple but very significant result was extended by Sommerfeld and Brillouin to show just how the wave advanced in the dispersive medium. ■

Summary — Inversion of Laplace Transform

- Direct use of tables, Table 15.2, and references; use of partial fractions (Section 15.8) and the operational theorems of Table 15.1.
- Bromwich integral, Eq. (15.212), and the calculus of residues.
- Numerical inversion, see the Additional Readings.

[24]Equation (15.225) follows rigorously from the theory of anomalous dispersion. See also the Kronig–Kramers optical dispersion relations of Section 7.2.

Table 15.1 Laplace Transform Operations

	Operations	Equation
1. Laplace transform	$f(s)=\mathcal{L}\{F(t)\}=\int_0^\infty e^{-st}F(t)\,dt$	(15.99)
2. Transform of derivative	$sf(s)-F(+0)=\mathcal{L}\{F'(t)\}$	(15.123)
	$s^2f(s)-sF(+0)-F'(+0)=\mathcal{L}\{F''(t)\}$	(15.124)
3. Transform of integral	$\frac{1}{s}f(s)=\mathcal{L}\left\{\int_0^t F(x)\,dx\right\}$	(Exercise 15.11.1)
4. Substitution	$f(s-a)=\mathcal{L}\{e^{at}F(t)\}$	(15.152)
5. Translation	$e^{-bs}f(s)=\mathcal{L}\{F(t-b)\}$	(15.164)
6. Derivative of transform	$f^{(n)}(s)=\mathcal{L}\{(-t)^nF(t)\}$	(15.173)
7. Integral of transform	$\int_s^\infty f(x)\,dx=\mathcal{L}\left\{\frac{F(t)}{t}\right\}$	(15.189)
8. Convolution	$f_1(s)f_2(s)=\mathcal{L}\left\{\int_0^t F_1(t-z)F_2(z)\,dz\right\}$	(15.193)
9. Inverse transform, Bromwich integral	$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}f(s)\,ds=F(t)$	(15.212)

Exercises

15.12.1 Derive the Bromwich integral from Cauchy's integral formula.
Hint. Apply the inverse transform $\mathcal{L}^{-1}$ to

$$f(s)=\frac{1}{2\pi i}\lim_{\alpha\to\infty}\int_{\gamma-i\alpha}^{\gamma+i\alpha}\frac{f(z)}{s-z}dz,$$

where $f(z)$ is analytic for $\Re(z)\ge\gamma$.

15.12.2 Starting with

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}f(s)\,ds,$$

show that by introducing

$$f(s)=\int_0^\infty e^{-sz}F(z)\,dz,$$

we can convert one integral into the Fourier representation of a Dirac delta function. From this derive the inverse Laplace transform.

15.12.3 Derive the Laplace transformation convolution theorem by use of the Bromwich integral.

15.12.4 Find

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

(a) by a partial fraction expansion.
(b) Repeat, using the Bromwich integral.

Table 15.2 Laplace Transforms

$f(s)$	$F(t)$	Limitation	Equation
1. 1	$\delta(t)$	Singularity at $+0$	(15.141)
2. $\frac{1}{s}$	1	$s > 0$	(15.102)
3. $\frac{n!}{s^{n+1}}$	t^n	$s > 0$ $n > -1$	(15.108)
4. $\frac{1}{s-k}$	e^{kt}	$s > k$	(15.103)
5. $\frac{1}{(s-k)^2}$	te^{kt}	$s > k$	(15.175)
6. $\frac{s}{s^2-k^2}$	$\cosh kt$	$s > k$	(15.105)
7. $\frac{k}{s^2-k^2}$	$\sinh kt$	$s > k$	(15.105)
8. $\frac{s}{s^2+k^2}$	$\cos kt$	$s > 0$	(15.107)
9. $\frac{k}{s^2+k^2}$	$\sin kt$	$s > 0$	(15.107)
10. $\frac{s-a}{(s-a)^2+k^2}$	$e^{at}\cos kt$	$s > a$	(15.153)
11. $\frac{k}{(s-a)^2+k^2}$	$e^{at}\sin kt$	$s > a$	(15.153)
12. $\frac{s^2-k^2}{(s^2+k^2)^2}$	$t\cos kt$	$s > 0$	(Exercise 15.10.19)
13. $\frac{2ks}{(s^2+k^2)^2}$	$t\sin kt$	$s > 0$	(Exercise 15.10.19)
14. $(s^2+a^2)^{-1/2}$	$J_0(at)$	$s > 0$	(15.185)
15. $(s^2-a^2)^{-1/2}$	$I_0(at)$	$s > a$	(Exercise 15.10.9)
16. $\frac{1}{a}\cot^{-1}\left(\frac{s}{a}\right)$	$j_0(at)$	$s > 0$	(Exercise 15.10.10)
17. $\left.\begin{array}{l} \frac{1}{2a}\ln\frac{s+a}{s-a} \\ \frac{1}{a}\coth^{-1}\left(\frac{s}{a}\right) \end{array}\right\}$	$i_0(at)$	$s > a$	(Exercise 15.10.10)
18. $\frac{(s-a)^n}{s^{n+1}}$	$L_n(at)$	$s > 0$	(Exercise 15.10.12)
19. $\frac{1}{s}\ln(s+1)$	$E_1(x) = -Ei(-x)$	$s > 0$	(Exercise 15.10.13)
20. $\frac{\ln s}{s}$	$-\ln t - \gamma$	$s > 0$	(Exercise 15.12.9)

A more extensive table of Laplace transforms appears in Chapter 29 of AMS-55 (see footnote 4 in Chapter 5 for the reference).

15.12.5 Find

$$\mathcal{L}^{-1}\left\{\frac{k^2}{s(s^2+k^2)}\right\}$$

(a) by using a partial fraction expansion.
(b) Repeat using the convolution theorem.
(c) Repeat using the Bromwich integral.

ANS. $F(t) = 1 - \cos kt$.

15.12.6 Use the Bromwich integral to find the function whose transform is $f(s) = s^{-1/2}$. Note that $f(s)$ has a branch point at $s = 0$. The negative x-axis may be taken as a cut line.

ANS. $F(t) = (\pi t)^{-1/2}$.

15.12.7 Show that

$$\mathcal{L}^{-1}\left\{\left(s^2+1\right)^{-1/2}\right\} = J_0(t)$$

by evaluation of the Bromwich integral.
Hint. Convert your Bromwich integral into an integral representation of $J_0(t)$. Figure 15.20 shows a possible contour.

15.12.8 Evaluate the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\left(s^2-a^2\right)^{-1/2}\right\}$$

by each of the following methods:

(a) Expansion in a series and term-by-term inversion.
(b) Direct evaluation of the Bromwich integral.
(c) Change of variable in the Bromwich integral: $s = (a/2)(z + z^{-1})$.

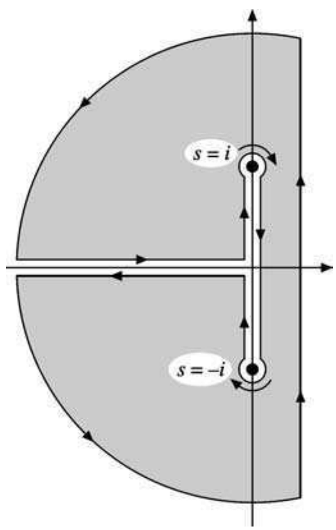

FIGURE 15.20 A possible contour for the inversion of $J_0(t)$.

15.12.9 Show that

$$\mathcal{L}^{-1}\left\{\frac{\ln s}{s}\right\} = -\ln t - \gamma,$$

where $\gamma = 0.5772\ldots$, the Euler–Mascheroni constant.

15.12.10 Evaluate the Bromwich integral for

$$f(s) = \frac{s}{(s^2 + a^2)^2}.$$

15.12.11 **Heaviside expansion theorem**. If the transform $f(s)$ may be written as a ratio

$$f(s) = \frac{g(s)}{h(s)},$$

where $g(s)$ and $h(s)$ are analytic functions, $h(s)$ having simple, isolated zeros at $s = s_i$, show that

$$F(t) = \mathcal{L}^{-1}\left\{\frac{g(s)}{h(s)}\right\} = \sum_i \frac{g(s_i)}{h'(s_i)} e^{s_i t}.$$

Hint. See Exercise 6.6.2.

15.12.12 Using the Bromwich integral, invert $f(s) = s^{-2}e^{-ks}$. Express $F(t) = \mathcal{L}^{-1}\{f(s)\}$ in terms of the (shifted) unit step function $u(t-k)$.

ANS. $F(t) = (t-k)u(t-k)$.

15.12.13 You have a Laplace transform:

$$f(s) = \frac{1}{(s+a)(s+b)}, \qquad a \neq b.$$

Invert this transform by each of three methods:

(a) Partial fractions and use of tables.
(b) Convolution theorem.
(c) Bromwich integral.

ANS. $F(t) = \dfrac{e^{-bt} - e^{-at}}{a-b}$, $a \neq b$.

Additional Readings

Champeney, D. C., *Fourier Transforms and Their Physical Applications*. New York: Academic Press (1973). Fourier transforms are developed in a careful, easy-to-follow manner. Approximately 60% of the book is devoted to applications of interest in physics and engineering.

Erdelyi, A.,W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, 2 vols. New York: McGraw–Hill (1954). This text contains extensive tables of Fourier sine, cosine, and exponential transforms, Laplace and inverse Laplace transforms, Mellin and inverse Mellin transforms, Hankel transforms, and other, more specialized integral transforms.

Hanna, J. R., *Fourier Series and Integrals of Boundary Value Problems*. Somerset, NJ: Wiley (1990). This book is a broad treatment of the Fourier solution of boundary value problems. The concepts of convergence and completeness are given careful attention.

Jeffreys, H., and B. S. Jeffreys, *Methods of Mathematical Physics*, 3rd ed. Cambridge, UK: Cambridge University Press (1972).

Krylov, V. I., and N. S. Skoblya, *Handbook of Numerical Inversion of Laplace* Transform. Jerusalem: Israel Program for Scientific Translations (1969).

Lepage, W. R., *Complex Variables and the Laplace Transform for Engineers*. New York: McGraw-Hill (1961); New York: Dover (1980). A complex variable analysis that is carefully developed and then applied to Fourier and Laplace transforms. It is written to be read by students, but intended for the serious student.

McCollum, P. A., and B. F. Brown, *Laplace Transform Tables and Theorems*. New York: Holt, Rinehart and Winston (1965).

Miles, J. W., *Integral Transforms in Applied Mathematics*. Cambridge, UK: Cambridge University Press (1971). This is a brief but interesting and useful treatment for the advanced undergraduate. It emphasizes applications rather than abstract mathematical theory.

Papoulis, A., *The Fourier Integral and Its Applications*. New York: McGraw-Hill (1962). This is a rigorous development of Fourier and Laplace transforms and has extensive applications in science and engineering.

Roberts, G. E., and H. Kaufman, *Table of Laplace Transforms*. Philadelphia: Saunders (1966).

Sneddon, I. N., *Fourier Transforms*. New York: McGraw-Hill (1951), reprinted, Dover (1995). A detailed comprehensive treatment, this book is loaded with applications to a wide variety of fields of modern and classical physics.

Sneddon, I. H., *The Use of Integral Transforms*. New York: McGraw-Hill (1972). Written for students in science and engineering in terms they can understand, this book covers all the integral transforms mentioned in this chapter as well as in several others. Many applications are included.

Van der Pol, B., and H. Bremmer, *Operational Calculus Based on the Two-sided Laplace Integral*, 3rd ed. Cambridge, UK: Cambridge University Press (1987). Here is a development based on the integral range $-\infty$ to $+\infty$, rather than the useful 0 to ∞. Chapter V contains a detailed study of the Dirac delta function (impulse function).

Wolf, K. B., *Integral Transforms in Science and Engineering*. New York: Plenum Press (1979). This book is a very comprehensive treatment of integral transforms and their applications.